Eur. J. Phys. **27** (2006) 29–47 doi:10.1088/0143-0807/27/1/004

# Learning about spin-one-half fields

**Peter Cahill**[1] **and Kevin Cahill**[2]

[1] Department of Mechanical Engineering, University of New Mexico, Albuquerque, NM 87131, USA
[2] New Mexico Center for Particle Physics, Department of Physics and Astronomy, University of New Mexico, Albuquerque, NM 87131, USA

E-mail: pedo@unm.edu and cahill@unm.edu

Received 14 March 2005, in final form 29 September 2005
Published 9 November 2005
Online at stacks.iop.org/EJP/27/29

**Abstract**
It is hard to understand spin-one-half fields without reading Weinberg. This paper is a pedagogical footnote to his formalism with an emphasis on the boost matrix, spinors and Majorana fields.

## 1. Introduction

The construction of Majorana and Dirac fields confuses many students year after year. Those who are not confused are likely to be those who have learned the subject from Weinberg's papers [1–3] or from volume one [4] of his treatise on quantum field theory. In that book, he describes how states respond to Poincaré transformations and infers how creation and annihilation operators transform. Then he shows that fields, which are linear combinations of these operators, transform suitably if their coefficients, the spinors, are related to suitable zero-momentum spinors by a 'standard boost' matrix $D(L(p))$. We cannot improve upon Weinberg's treatment, but we can add to it—the present paper is a long pedagogical footnote to section 5.5 of his book [4].

In section 2, we recall his construction of spinors from the Dirac representation $D(L(p))$ of the standard boost $L(p)$. In section 3, we derive for the matrix $D(L(p))$ the simple expression

$$D(L(p)) = \frac{(m + p_a\gamma^a\gamma^0)}{\sqrt{2m(p^0 + m)}} \tag{1}$$

which leads directly to useful matrix formulas for the spinors $u(\mathbf{p}, s)$ and $v(\mathbf{p}, s)$. We suggest classroom use of this expression and these matrix formulas. They imply that the spinors satisfy the Dirac equation in momentum space and the Majorana conditions, and they simplify the evaluation of spin sums.

In section 4, we construct the Majorana field $\chi(x)$ and show that it satisfies the Majorana condition. We use spin sums to compute its propagator and its anti-commutator with itself and its adjoint. We also relate it to a four-component, anti-commuting scalar-like field. In section 5, we construct the Dirac field $\psi(x)$ from two Majorana fields that describe particles of the same mass, and we show that its anti-particle field is $\psi_c = \gamma^2 \psi^*$ because its constituent Majorana fields satisfy the Majorana condition. We discuss the propagator and the causality and helicity properties of the Dirac field, which we relate to a complex, four-component, anti-commuting scalar-like field. Finally, in section 6, we apply our Majorana formulas to the Wess–Zumino model and its supercharges. The lessons of sections 4–6 are appropriate for classroom use.

## 2. Relativity, causality, Majorana fields and spinors

In this section, we present a distillation for Majorana fields of Weinberg's discussion [4] of spin-one-half fields. This section provides the context of this paper.

The particle-annihilation

$$\chi_b^+(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \sum_s u_b(\mathbf{p}, s)\, a(\mathbf{p}, s)\, e^{ipx} \tag{2}$$

and particle-creation fields

$$\chi_b^-(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \sum_s v_b(\mathbf{p}, s)\, a^\dagger(\mathbf{p}, s)\, e^{-ipx} \tag{3}$$

of a spin-one-half Majorana $\chi(x)$ field will transform suitably under Poincaré transformations

$$U(\Lambda, a)\chi_a^\pm(x)U^{-1}(\Lambda, a) = \sum_b D_{ab}(\Lambda^{-1})\chi_b^\pm(\Lambda x + a) \tag{4}$$

if the spinors $u(\mathbf{p}, s)$ and $v(\mathbf{p}, s)$ are related to suitable zero-momentum spinors $u(\mathbf{0}, s)$ and $v(\mathbf{0}, s)$ by a matrix $D_{ab}(L(p))$

$$\begin{aligned} u(\mathbf{p}, s) &= \sqrt{m/p^0}\, D(L(p))\, u(\mathbf{0}, s) \\ v(\mathbf{p}, s) &= \sqrt{m/p^0}\, D(L(p))\, v(\mathbf{0}, s) \end{aligned} \tag{5}$$

that represents the standard boost

$$L(p)_b{}^0 = -p_b/m \tag{6}$$

that takes $(m, \mathbf{0})$ into $(p^0, \mathbf{p})$ in the Dirac representation of the Lorentz group, as explained in section 5.4 of [4]. The $4 \times 4$ matrices $D(\Lambda)$ of this representation are exponentials

$$D(\Lambda) = e^{\frac{i}{2}\omega_{ab}\mathcal{J}^{ab}} \tag{7}$$

of generators of the Lorentz group

$$\mathcal{J}^{ab} = -\frac{i}{4}[\gamma^a, \gamma^b] \tag{8}$$

in which the gamma matrices satisfy the anti-commutation relations

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab} \tag{9}$$

where $\eta$ is the flat spacetime metric $\eta = \mathrm{diag}(-1, 1, 1, 1)$. In the Dirac representation, the gamma matrices $\gamma^a$ transform as a vector

$$D(\Lambda)\gamma^a D^{-1}(\Lambda) = \Lambda_b{}^a \gamma^b. \tag{10}$$

The fields $\chi^\pm(x)$ will transform suitably under parity if the zero-momentum spinors $u(\mathbf{0}, s)$ and $v(\mathbf{0}, s)$ are eigenstates of $i\gamma^0$ with eigenvalues $\pm 1$.

Although the fields $\chi_b^{\pm}(x)$ do not commute or anti-commute with their adjoints, the Majorana field that is their sum

$$\chi_b(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \sum_s [u_b(\mathbf{p}, s)\, a(\mathbf{p}, s)\, e^{ipx} + v_b(\mathbf{p}, s)\, a^\dagger(\mathbf{p}, s)\, e^{-ipx}] \tag{11}$$

will anti-commute at space-like separations both with itself and its adjoint if the operators $a\big(\mathbf{p}, \pm\frac{1}{2}\big)$ and $a^\dagger\big(\mathbf{p}, \pm\frac{1}{2}\big)$ obey the anti-commutation relations

$$\{a(\mathbf{p}, s), a^\dagger(\mathbf{p}', s')\} = \delta_{ss'}\, \delta(\mathbf{p} - \mathbf{p}') \tag{12}$$

and

$$\{a(\mathbf{p}, s), a(\mathbf{p}', s')\} = 0 \tag{13}$$

and if the zero-momentum spinors are eigenstates of $i\gamma^0$ with opposite eigenvalues (1 and $-1$, or $-1$ and 1) [4]. The usual choice is

$$i\gamma^0 u(\mathbf{0}, s) = u(\mathbf{0}, s), \qquad i\gamma^0 v(\mathbf{0}, s) = -\, v(\mathbf{0}, s). \tag{14}$$

If the gamma matrices are taken to be

$$\gamma^k = -i \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix} \qquad k = 1, 2, 3 \tag{15}$$

and

$$\gamma^0 = -i\beta = -i \begin{pmatrix} \mathbf{0} & I \\ I & \mathbf{0} \end{pmatrix}, \tag{16}$$

then a natural choice [4] for the zero-momentum spinors is

$$u\left(\mathbf{0}, \frac{1}{2}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad u\left(\mathbf{0}, -\frac{1}{2}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \tag{17}$$

and

$$v\left(\mathbf{0}, \frac{1}{2}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \qquad v\left(\mathbf{0}, -\frac{1}{2}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{18}$$

which incidentally satisfy the Majorana conditions

$$\begin{aligned} u(\mathbf{0}, s) &= \gamma^2 v^*(\mathbf{0}, s) = \gamma^2 v(\mathbf{0}, s) \\ v(\mathbf{0}, s) &= \gamma^2 u^*(\mathbf{0}, s) = \gamma^2 u(\mathbf{0}, s). \end{aligned} \tag{19}$$

The spatial gamma matrices are Hermitian, and the temporal one is anti-Hermitian

$$(\gamma^k)^\dagger = \gamma^k \qquad \text{and} \qquad (\gamma^0)^\dagger = -\gamma^0. \tag{20}$$

The gamma matrices of even index are symmetric, and those of odd index are anti-symmetric

$$(\gamma^a)^\top = (-1)^a\, \gamma^a = -\gamma^0\gamma^2\gamma^a\gamma^0\gamma^2. \tag{21}$$

The matrix $\gamma_5$ is

$$\gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{22}$$

With a different set of $\gamma$-matrices $\gamma^{a'} = S\gamma^a S^{-1}$, the fields and spinors should be multiplied from the left by the matrix $S$. A nice feature of the chosen $\gamma$-matrices (15) and (16) is that the Lorentz generators $\mathcal{J}^{ab}$ are block diagonal, as is formula (1) for the standard boost

$$D(L(p)) = \frac{1}{\sqrt{2m(p^0+m)}} \begin{pmatrix} p^0 + m - \mathbf{p}\cdot\vec{\sigma} & 0 \\ 0 & p^0 + m + \mathbf{p}\cdot\vec{\sigma} \end{pmatrix} \tag{23}$$

in the Dirac representation of the Lorentz group. A derivation of formula (1) for the standard boost is given in the next section.

## 3. Spinors

In this section, we will find useful matrix formulas for the spinors $u(\mathbf{p}, s)$ and $v(\mathbf{p}, s)$ from their definitions (5). The key step will be the explicit evaluation of the standard boost $D(L(p))$. We then use these matrix formulas to evaluate spin sums and to show that the spinors obey Majorana conditions and the Dirac equation in momentum space.

The standard boost $D(L(p))$ is [4]

$$D(L(p)) = D(R(\hat{\mathbf{p}})D(B(|\mathbf{p}|))D(R^{-1}(\hat{\mathbf{p}}) \tag{24}$$

where $B(|\mathbf{p}|)$ is a boost in the three-direction, and $R(\hat{\mathbf{p}})$ is a particular rotation that takes the three-axis into the direction $\hat{\mathbf{p}}$. Thus the standard boost is a boost in the direction $\hat{\mathbf{p}}$ that takes the four-vector $(m, \mathbf{0})$ to $p$. The generator of such boosts is proportional to $4\mathrm{i}\mathcal{J}^{i0}p^i = [\gamma^i, \gamma^0]p^i$ and so to $\hat{\mathbf{p}}\cdot\vec{\gamma}\gamma^0$. So the standard boost $D(L(p))$ is

$$D(L(p)) = \mathrm{e}^{\alpha\hat{\mathbf{p}}\cdot\vec{\gamma}\gamma^0} \tag{25}$$

in which $\alpha$ is a parameter whose value is constrained by the requirements (6) and (10)

$$\begin{aligned} D(L(p))\gamma^0 D^{-1}(L(p)) &= \mathrm{e}^{\alpha\hat{\mathbf{p}}\cdot\vec{\gamma}\gamma^0}\gamma^0\,\mathrm{e}^{-\alpha\hat{\mathbf{p}}\cdot\vec{\gamma}\gamma^0} = L(p)_b{}^0\gamma^b \\ &= -p_b\gamma^b/m = -p_b\gamma^b/m. \end{aligned} \tag{26}$$

Because $(\alpha\hat{\mathbf{p}}\cdot\vec{\gamma}\gamma^0)^2 = \alpha^2$, the gamma matrix $\gamma^0$ is transformed to

$$\begin{aligned} \mathrm{e}^{\alpha\hat{\mathbf{p}}\cdot\vec{\gamma}\gamma^0}\gamma^0\,\mathrm{e}^{-\alpha\hat{\mathbf{p}}\cdot\vec{\gamma}\gamma^0} &= \mathrm{e}^{2\alpha\hat{\mathbf{p}}\cdot\vec{\gamma}\gamma^0}\gamma^0 \\ &= (\cosh 2\alpha + \hat{\mathbf{p}}\cdot\vec{\gamma}\gamma^0\sinh 2\alpha)\gamma^0 \\ &= \gamma^0\cosh 2\alpha - \hat{\mathbf{p}}\cdot\vec{\gamma}\sinh 2\alpha = -p_b\gamma^b/m \end{aligned} \tag{27}$$

by the preceding equation. So $\cosh 2\alpha = p^0/m$, whence

$$\cosh\alpha = \sqrt{\frac{p^0+m}{2m}} \tag{28}$$

and

$$\sinh\alpha = \sqrt{\frac{p^0-m}{2m}}. \tag{29}$$

Thus the standard boost $D(L(p))$ is

$$\begin{aligned} D(L(p)) = \mathrm{e}^{\alpha\hat{\mathbf{p}}\cdot\vec{\gamma}\gamma^0} &= \cosh\alpha + \hat{\mathbf{p}}\cdot\vec{\gamma}\gamma^0\sinh\alpha \\ &= \sqrt{\frac{p^0+m}{2m}} + \hat{\mathbf{p}}\cdot\vec{\gamma}\gamma^0\sqrt{\frac{p^0-m}{2m}} \\ &= \frac{(p^0+m+\mathbf{p}\cdot\vec{\gamma}\gamma^0)}{\sqrt{2m(p^0+m)}} \end{aligned} \tag{30}$$

or since $(\gamma^0)^2 = -1$

$$D(L(p)) = \frac{(m + p_a \gamma^a \gamma^0)}{\sqrt{2m(p^0 + m)}}. \tag{31}$$

This simple, explicit formula for the standard boost leads directly to expressions for the spinors and their spin sums, and so we recommend its use in classrooms. It is true for any choice of gamma matrices that satisfy $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$ with $\eta = \mathrm{diag}(-1, 1, 1, 1)$.

By its definition (5) in terms of the standard boost (31), the spinor $u(\mathbf{p}, s)$ is

$$u(\mathbf{p}, s) = \sqrt{\frac{m}{p^0}}\, D(L(p))\, u(\mathbf{0}, s) = \frac{(m + p_a \gamma^a \gamma^0)}{\sqrt{2p^0(p^0 + m)}}\, u(\mathbf{0}, s) \tag{32}$$

or

$$u(\mathbf{p}, s) = \frac{(m - \mathrm{i} p_a \gamma^a)}{\sqrt{2p^0(p^0 + m)}}\, u(\mathbf{0}, s) \tag{33}$$

since $\gamma^0 u(\mathbf{0}, s) = -\mathrm{i} u(\mathbf{0}, s)$ by (14).

Similarly, by its definition (5) in terms of the standard boost (31), the spinor $v(\mathbf{p}, s)$ is

$$v(\mathbf{p}, s) = \sqrt{\frac{m}{p^0}}\, D(L(p))\, v(\mathbf{0}, s) = \frac{(m + p_a \gamma^a \gamma^0)}{\sqrt{2p^0(p^0 + m)}}\, v(\mathbf{0}, s) \tag{34}$$

or

$$v(\mathbf{p}, s) = \frac{(m + \mathrm{i} p_a \gamma^a)}{\sqrt{2p^0(p^0 + m)}}\, v(\mathbf{0}, s) \tag{35}$$

since $\gamma^0 v(\mathbf{0}, s) = \mathrm{i} v(\mathbf{0}, s)$ by (14).

These matrix formulas (33) and (35) for the spinors $u(\mathbf{p}, s)$ and $v(\mathbf{p}, s)$ follow from the definition (5) of the spinors in terms of the standard boost, the expression (31) for that boost, and from the eigenvalue equations (14). They are quite independent of the choice of gamma matrices. For a different set of $\gamma$-matrices $\gamma'^a = S\gamma^a S^{-1}$, one merely multiplies the $\mathbf{p} = 0$ spinors (17) and (18) and the spinors $u(\mathbf{p}, s)$ and $v(\mathbf{p}, s)$ of equations (33) and (35) by the matrix $S$, an operation under which the conditions (14) and (19) are covariant. Our derivation of the boost formula (31) and of the matrix formulas (33) and (35) and of some of their consequences are the main content of this paper.

The adjoints $\overline{u}(\mathbf{p}, s) = \mathrm{i} u^\dagger(\mathbf{p}, s)\gamma^0$ and $\overline{v}(\mathbf{p}, s) = \mathrm{i} v^\dagger(\mathbf{p}, s)\gamma^0$ of the spinors (33) and (35) are

$$\overline{u}(\mathbf{p}, s) = \overline{u}(\mathbf{0}, s)\frac{(m - \mathrm{i} p_a \gamma^a)}{\sqrt{2p^0(p^0 + m)}} \quad \text{and} \quad \overline{v}(\mathbf{p}, s) = \overline{v}(\mathbf{0}, s)\frac{(m + \mathrm{i} p_a \gamma^a)}{\sqrt{2p^0(p^0 + m)}}. \tag{36}$$

The transposes of the spinors (33) and (35) are by (21)

$$u^\top(\mathbf{p}, s) = u^\top(\mathbf{0}, s)\frac{\gamma^0\gamma^2(m + \mathrm{i} p_a \gamma^a)\gamma^0\gamma^2}{\sqrt{2p^0(p^0 + m)}} \tag{37}$$

and

$$v^\top(\mathbf{p}, s) = v^\top(\mathbf{0}, s)\frac{\gamma^0\gamma^2(m - \mathrm{i} p_a \gamma^a)\gamma^0\gamma^2}{\sqrt{2p^0(p^0 + m)}}. \tag{38}$$

### 3.1. Dirac equation

These matrix formulas (33) and (35) imply that the spinors $u(\mathbf{p}, s)$ and $v(\mathbf{p}, s)$ are eigenvectors of $-\mathrm{i}p_a\gamma^a$ with eigenvalues $\pm m$:

$$(\mathrm{i}p_a\gamma^a + m)u(\mathbf{p}, s) = (\mathrm{i}p_a\gamma^a + m)\frac{(m - \mathrm{i}p_b\gamma^b)\, u(\mathbf{0}, s)}{\sqrt{2p^0(p^0 + m)}} = \frac{(m^2 + p^a p_a)\, u(\mathbf{0}, s)}{\sqrt{2p^0(p^0 + m)}} = 0 \tag{39}$$

and

$$(m - \mathrm{i}p_a\gamma^a)v(\mathbf{p}, s) = (m - \mathrm{i}p_a\gamma^a)\frac{(m + \mathrm{i}p_a\gamma^a)\, v(\mathbf{0}, s)}{\sqrt{2p^0(p^0 + m)}} = \frac{(m^2 + p^a p_a)\, v(\mathbf{0}, s)}{\sqrt{2p^0(p^0 + m)}} = 0. \tag{40}$$

That is, they satisfy the Dirac equation in momentum space.

Thus the Majorana field (11) satisfies the Dirac equation

$$(\gamma^a\partial_a + m)\chi(x) = 0 \tag{41}$$

in position space.

The adjoint spinors (36) also satisfy the Dirac equation in momentum space

$$\overline{u}(\mathbf{p}, s)(m + \mathrm{i}p_a\gamma^a) = 0 \qquad \text{and} \qquad \overline{v}(\mathbf{p}, s)(m - \mathrm{i}p_a\gamma^a) = 0. \tag{42}$$

### 3.2. Majorana condition

The spinors (33) and (35) satisfy the Majorana conditions

$$u(\mathbf{p}, s) = \gamma^2 v^*(\mathbf{p}, s) \qquad \text{and} \qquad v(\mathbf{p}, s) = \gamma^2 u^*(\mathbf{p}, s) \tag{43}$$

as they do (19) at $\mathbf{p} = \mathbf{0}$.

### 3.3. Spin sums

The matrix formulas (33) and (35) for the spinors $u(\mathbf{p}, s)$ and $v(\mathbf{p}, s)$ simplify the evaluation of spin sums. We start with the spin sums over the $\mathbf{p} = 0$ spinors (17) and (18),

$$\sum_s u(\mathbf{0}, s)\, \overline{u}(\mathbf{0}, s) = \frac{1}{2}(\mathrm{i}\gamma^0 + I) \tag{44}$$

and

$$\sum_s v(\mathbf{0}, s)\, \overline{v}(\mathbf{0}, s) = \frac{1}{2}(\mathrm{i}\gamma^0 - I) \tag{45}$$

as well as

$$\sum_s u(\mathbf{0}, s)\, v^\top(\mathbf{0}, s) = \frac{1}{2}(I + \mathrm{i}\gamma^0)\gamma^2 \tag{46}$$

and

$$\sum_s v(\mathbf{0}, s)\, u^\top(\mathbf{0}, s) = \frac{1}{2}(I - \mathrm{i}\gamma^0)\gamma^2. \tag{47}$$

Thus by (33), (36) and (44), the spin sum of the outer products $u(\mathbf{p}, s)\overline{u}(\mathbf{p}, s)$ is

$$\sum_s u(\mathbf{p}, s)\overline{u}(\mathbf{p}, s) = \frac{(m - \mathrm{i}p_a\gamma^a)\,(\mathrm{i}\gamma^0 + I)(m - \mathrm{i}p_b\gamma^b)}{4p^0(p^0 + m)} \tag{48}$$

which the gamma-matrix algebra (9) and the mass-shell relation $p^2 = m^2$ reduce to

$$\sum_s u(\mathbf{p}, s)\overline{u}(\mathbf{p}, s) = \frac{m - \mathrm{i}p_a\gamma^a}{2p^0}. \tag{49}$$

The spin sum with $u^\dagger(\mathbf{p}, s) = \mathrm{i}\overline{u}(\mathbf{p}, s)\gamma^0$ is

$$\sum_s u(\mathbf{p}, s)\, u^\dagger(\mathbf{p}, s) = \frac{(\mathrm{i}m + p_a\gamma^a)\gamma^0}{2p^0}. \tag{50}$$

Similarly by (35), (36) and (45), the spin sum of the outer products $v(\mathbf{p}, s)\overline{v}(\mathbf{p}, s)$ is

$$\sum_s v(\mathbf{p}, s)\overline{v}(\mathbf{p}, s) = \frac{(m + \mathrm{i}p_a\gamma^a)\,(\mathrm{i}\gamma^0 - I)(m + \mathrm{i}\gamma^b p_b)}{4p^0(p^0 + m)} \tag{51}$$

which, differing from (48) as it does by $\mathrm{i} \to -\mathrm{i}$ and by an overall minus sign, is

$$\sum_s v(\mathbf{p}, s)\overline{v}(\mathbf{p}, s) = \frac{-m - \mathrm{i}p_a\gamma^a}{2p^0}. \tag{52}$$

The spin sum with $v^\dagger(\mathbf{p}, s) = \mathrm{i}\overline{v}(\mathbf{p}, s)\gamma^0$ is

$$\sum_s v(\mathbf{p}, s)v^\dagger(\mathbf{p}, s) = \frac{(p_a\gamma^a - \mathrm{i}m)\,\gamma^0}{2p^0}. \tag{53}$$

With a little more effort, one finds from (33), (46) and (38) the spin sum

$$\begin{aligned}\sum_s u(\mathbf{p}, s)\, v^\top(\mathbf{p}, s) &= \frac{(m - \mathrm{i}p_a\gamma^a)(1 + \mathrm{i}\gamma^0)\gamma^2\gamma^0\gamma^2(m - \mathrm{i}p_b\gamma^b)\gamma^0\gamma^2}{4p^0(p^0 + m)}\\ &= \frac{(\mathrm{i}m + p_a\gamma^a)\gamma^0\gamma^2}{2p^0}.\end{aligned} \tag{54}$$

Similarly, the $vu^\top$ spin sum follows from (35), (47) and (37),

$$\begin{aligned}\sum_s v(\mathbf{p}, s)\, u^\top(\mathbf{p}, s) &= \frac{(m + \mathrm{i}p_a\gamma^a)(1 - \mathrm{i}\gamma^0)\gamma^2\gamma^0\gamma^2(m + \mathrm{i}p_b\gamma^b)\gamma^0\gamma^2}{4p^0(p^0 + m)}\\ &= \frac{(-\mathrm{i}m + p_a\gamma^a)\gamma^0\gamma^2}{2p^0}\end{aligned} \tag{55}$$

and so differs from (54) by $\mathrm{i} \to -\mathrm{i}$, as it must.

### *3.4. Inner products of spinors*

The matrix formula (33) for the spinor $u(\mathbf{p}, s)$, the behaviour (20) of the $\gamma$-matrices under Hermitian conjugation and the eigenvalue relation $\mathrm{i}\gamma^0 u(\mathbf{0}, s) = u(\mathbf{0}, s)$ (14) imply that

$$u^\dagger(\mathbf{p}, s)\, u(\mathbf{p}, s') = \delta_{ss'}. \tag{56}$$

Similarly

$$v^\dagger(\mathbf{p}, s)\, v(\mathbf{p}, s') = \delta_{ss'} \tag{57}$$

and $u^\dagger(\mathbf{p}, s)v(-\mathbf{p}, s') = 0$.

By (14), the zero-momentum spinors $u(\mathbf{0}, s)$ and $v(\mathbf{0}, s')$ are eigenvectors of the Hermitian matrix $\mathrm{i}\gamma^0$ with eigenvalues $+1$ and $-1$. So these spinors are orthogonal, $u^\dagger(\mathbf{0}, s)v(\mathbf{0}, s') = 0$. Also, $\bar{u}(\mathbf{0}, s)u(\mathbf{0}, s') = u^\dagger(\mathbf{0}, s)\mathrm{i}\gamma^0 u(\mathbf{0}, s') = u^\dagger(\mathbf{0}, s)u(\mathbf{0}, s') = \delta_{ss'}$ and $\bar{v}(\mathbf{0}, s)v(\mathbf{0}, s') = v^\dagger(\mathbf{0}, s)\mathrm{i}\gamma^0 v(\mathbf{0}, s') = -v^\dagger(\mathbf{0}, s)v(\mathbf{0}, s') = -\delta_{ss'}$. Now $\mathrm{i}\gamma^0\vec{\gamma}u(\mathbf{0}, s) = -\vec{\gamma}\mathrm{i}\gamma^0 u(\mathbf{0}, s) = -\vec{\gamma}u(\mathbf{0}, s)$, and so the spinors $\vec{\gamma}u(\mathbf{0}, s)$ and $u(\mathbf{0}, s')$ are eigenvectors of the Hermitian matrix $\mathrm{i}\gamma^0$ with different eigenvalues and so must be orthogonal, $u^\dagger(\mathbf{0}, s)\vec{\gamma}u(\mathbf{0}, s') = 0$. Similarly, $v^\dagger(\mathbf{0}, s)\vec{\gamma}v(\mathbf{0}, s') = 0$. It follows therefore from the matrix formulas (33) for $u(\mathbf{p}, s)$ and (36) for $\bar{u}(\mathbf{p}, s')$ that

$$\bar{u}(\mathbf{p}, s)\, u(\mathbf{p}, s') = \frac{m}{p^0}\bar{u}(\mathbf{0}, s)\, u(\mathbf{0}, s') = \frac{m}{p^0}\,\delta_{ss'}. \tag{58}$$

Similarly

$$\bar{v}(\mathbf{p}, s)\, v(\mathbf{p}, s') = \frac{m}{p^0}\bar{v}(\mathbf{0}, s)\, v(\mathbf{0}, s') = -\frac{m}{p^0}\,\delta_{ss'}. \tag{59}$$

Since the spinors $u$ and $v$ obey the Dirac equation in momentum space (39) and (40), it follows that $\bar{u}(\mathbf{p}, s)\gamma^a(m + \mathrm{i}p'_b\gamma^b)u(\mathbf{p}', s') = 0$ and $\bar{u}(\mathbf{p}, s)(m + \mathrm{i}p_b\gamma^b)\gamma^a u(\mathbf{p}', s') = 0$. So

$$2m\bar{u}(\mathbf{p}, s)\gamma^a u(\mathbf{p}', s') = -\mathrm{i}\bar{u}(\mathbf{p}, s)(p_b\gamma^b\gamma^a + p'_b\gamma^a\gamma^b)u(\mathbf{p}', s') \tag{60}$$

and since $2\gamma^b\gamma^a = \{\gamma^b, \gamma^a\} + [\gamma^b, \gamma^a]$ one has

$$\bar{u}(\mathbf{p}, s)\gamma^a u(\mathbf{p}', s') = \frac{-\mathrm{i}}{2m}\,\bar{u}(\mathbf{p}, s)\left(p^a + p'^a + \tfrac{1}{2}(p_b - p'_b)\,[\gamma^b, \gamma^a]\right) u(\mathbf{p}', s') \tag{61}$$

which is Gordon's identity. Similarly,

$$\bar{v}(\mathbf{p}, s)\gamma^a v(\mathbf{p}', s') = \frac{\mathrm{i}}{2m}\,\bar{v}(\mathbf{p}, s)\left(p^a + p'^a + \tfrac{1}{2}(p_b - p'_b)\,[\gamma^b, \gamma^a]\right) v(\mathbf{p}', s'). \tag{62}$$

### *3.5. Explicit formulas for spinors*

The static spinors (17) and (18) and the matrix formulas (33) and (35) give explicit expressions for the spinors at arbitrary momentum $\mathbf{p}$:

$$u\left(\mathbf{p}, \frac{1}{2}\right) = \frac{1}{2\sqrt{p^0(p^0 + m)}}\begin{pmatrix} m + p^0 - p_3 \\ -p_1 - \mathrm{i}p_2 \\ m + p^0 + p_3 \\ p_1 + \mathrm{i}p_2 \end{pmatrix} \tag{63}$$

$$u\left(\mathbf{p}, -\frac{1}{2}\right) = \frac{1}{2\sqrt{p^0(p^0 + m)}}\begin{pmatrix} -p_1 + \mathrm{i}p_2 \\ m + p^0 + p_3 \\ p_1 - \mathrm{i}p_2 \\ m + p^0 - p_3 \end{pmatrix} \tag{64}$$

$$v\left(\mathbf{p}, \frac{1}{2}\right) = \frac{1}{2\sqrt{p^0(p^0 + m)}}\begin{pmatrix} -p_1 + \mathrm{i}p_2 \\ m + p^0 + p_3 \\ -p_1 + \mathrm{i}p_2 \\ -m - p^0 + p_3 \end{pmatrix} \tag{65}$$

$$v\left(\mathbf{p}, -\frac{1}{2}\right) = \frac{1}{2\sqrt{p^0(p^0 + m)}}\begin{pmatrix} -m - p^0 + p_3 \\ p_1 + \mathrm{i}p_2 \\ m + p^0 + p_3 \\ p_1 + \mathrm{i}p_2 \end{pmatrix} \tag{66}$$

which, as $\mathbf{p} \to \mathbf{0}$, reduce to the static spinors (17) and (18).

*3.6. Helicity*

For momenta in the $z$-direction and in the limit of small $m/p_3$, these formulas yield

$$u\left(\mathbf{p}, \frac{1}{2}\right) \approx \left(1 - \frac{m}{p}\right) \begin{pmatrix} m/(2p) \\ 0 \\ 1 + m/(2p) \\ 0 \end{pmatrix} \tag{67}$$

$$u\left(\mathbf{p}, -\frac{1}{2}\right) \approx \left(1 - \frac{m}{p}\right) \begin{pmatrix} 0 \\ 1 + m/(2p) \\ 0 \\ m/(2p) \end{pmatrix} \tag{68}$$

$$v\left(\mathbf{p}, \frac{1}{2}\right) \approx \left(1 - \frac{m}{p}\right) \begin{pmatrix} 0 \\ 1 + m/(2p) \\ 0 \\ -m/(2p) \end{pmatrix} \tag{69}$$

$$v\left(\mathbf{p}, -\frac{1}{2}\right) \approx \left(1 - \frac{m}{p}\right) \begin{pmatrix} -m/(2p) \\ 0 \\ 1 + m/(2p) \\ 0 \end{pmatrix} \tag{70}$$

in which $p = |\mathbf{p}| = p_3 \geqslant 0$.

## 4. Majorana field

In this section, we apply our spinor formulas to the Majorana field, which is simpler and more fundamental than the Dirac field.

In terms of the annihilation and creation operators (12) and (13) and the spinors (33) and (35), the Majorana field is

$$\chi_b(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \sum_s [u_b(\mathbf{p}, s)\, a(\mathbf{p}, s)\, e^{ipx} + v_b(\mathbf{p}, s)\, a^\dagger(\mathbf{p}, s)\, e^{-ipx}]. \tag{71}$$

It satisfies the Majorana condition

$$\chi(x) = \gamma^2 \chi^*(x) \tag{72}$$

because the spinors $u(\mathbf{p}, s)$ and $v(\mathbf{p}, s)$ do (43). It obeys the Dirac equation

$$(\gamma^a \partial_a + m)\chi(x) = 0 \tag{73}$$

because the spinors do so in momentum space (39) and (40).

An action density that leads to this Dirac equation is

$$\mathcal{L}_M = -\tfrac{1}{4}\bar{\chi}\gamma^a\partial_a\chi + \tfrac{1}{4}(\partial_a\bar{\chi})\gamma^a\chi - \tfrac{1}{2}m\bar{\chi}\chi \tag{74}$$

in which $-(m/2)\bar{\chi}\chi$ is a Majorana mass term. If there are several Majorana fields, then the symmetry $\bar{\chi}_i\chi_j = \bar{\chi}_j\chi_i = (\bar{\chi}_i\chi_j)^\dagger$ implies that the matrix of coefficients $m_{ij}$ is real and symmetric.

### 4.1. Helicity

In the limit $m/p_3 \to 0+$, we may infer from the spinor formulas (67)–(70) and from the form (71) of the Majorana field that its upper two components annihilate particles of negative helicity and create particles of positive helicity, while its lower two components annihilate particles of positive helicity and create particles of negative helicity.

More generally, from the explicit spinor formulas (63)–(66) and from (71), we may infer that particles created by the field $(1+\gamma_5)\chi$ are partially positively polarized, while those created by the field $\chi^\dagger(1+\gamma_5)$ are partially negatively polarized. The weak charged current selects these upper two components.

### 4.2. Causality

The spin sums (54) and (55) imply that the anti-commutator of two of its components is

$$\begin{aligned}\{\chi_a(x),\chi_b(y)\} &= \int \frac{d^3p}{(2\pi)^3}\sum_s (u_a(\mathbf{p},s)v_b(\mathbf{p},s)\,e^{ip(x-y)} + v_a(\mathbf{p},s)u_b(\mathbf{p},s)\,e^{-ip(x-y)}) \\ &= \int \frac{d^3p}{(2p^0)(2\pi)^3}[((im+\gamma^c p_c)\gamma^0\gamma^2)_{ab}\,e^{ip(x-y)} \\ &\quad + ((-im+\gamma^c p_c)\gamma^0\gamma^2)_{ab}\,e^{-ip(x-y)}] \\ &= i(m-\gamma^c\partial_c)_{ab}\Delta(x-y) \end{aligned} \tag{75}$$

in which $\Delta(x-y)$ is the Lorentz-invariant function

$$\Delta(x-y) = \int \frac{d^3p}{(2p^0)(2\pi)^3}(e^{ip(x-y)} - e^{-ip(x-y)}) \tag{76}$$

which vanishes at space-like separations. The equal-time anti-commutator is

$$\{\chi_a(x),\chi_b(y)\}|_{x^0=y^0} = \gamma^2_{ab}\,\delta(\mathbf{x}-\mathbf{y}). \tag{77}$$

Similarly, the spin sums (49) and (52) imply that

$$\{\chi_a(x),\overline{\chi_b}(y)\} = (m-\gamma^c\partial_c)_{ab}\,\Delta(x-y) \tag{78}$$

so that the equal-time anti-commutator is

$$\{\chi_a(x),\overline{\chi_b}(y)\}|_{x^0=y^0} = i\gamma^0_{ab}\,\delta(\mathbf{x}-\mathbf{y}) \tag{79}$$

or equivalently

$$\left\{\chi_a(x),\chi_b^\dagger(y)\right\}\Big|_{x^0=y^0} = \delta_{ab}\,\delta(\mathbf{x}-\mathbf{y}). \tag{80}$$

### 4.3. Propagator

In the usual way [4], by using the spin sums (49) and (52), one may evaluate the propagator

$$\langle 0|T\,\{\chi_a(x)\bar\chi_b(y)\}\,|0\rangle = \int \frac{d^4p}{(2\pi)^4}\frac{\gamma^c p_c + im}{p^2+m^2-i\epsilon}\,e^{ip(x-y)}. \tag{81}$$

### 4.4. Scalar-like field

The matrix formulas (33) and (35) imply that we may define the Majorana field $\chi(x)$ in terms of the simpler field

$$\phi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0(p^0+m)}} \sum_{s=-\frac{1}{2}}^{\frac{1}{2}} [u(\mathbf{0},s)\, a(\mathbf{p},s)\, e^{ipx} + v(\mathbf{0},s)\, a^\dagger(\mathbf{p},s)\, e^{-ipx}] \tag{82}$$

as

$$\chi(x) = (m - \gamma^a \partial_a)\phi(x). \tag{83}$$

Because the $\mathbf{p} = \mathbf{0}$ spinors $u(\mathbf{0}, s)$ and $v(\mathbf{0}, s)$ are independent of momentum, the field $\phi(x)$ is like a scalar field—or like four scalar fields.

Since $m^2 + p^2 = m^2 + \mathbf{p}^2 - (p^0)^2 = 0$, the scalar-like field $\phi(x)$ satisfies the Klein–Gordon equation

$$\left(m^2 + \partial_0^2 - \nabla^2\right)\phi(x) = (m^2 - \eta^{ab}\partial_a\partial_b)\phi(x) = 0. \tag{84}$$

The derivative formula (83) for the Majorana field $\chi(x)$ implies that it satisfies the Dirac equation

$$\begin{aligned}(\gamma^a\partial_a + m)\chi(x) &= (\gamma^a\partial_a + m)(m - \gamma^a\partial_a)\phi(x) \\ &= (m^2 - \gamma^a\gamma^b\partial_a\partial_b)\phi(x) \\ &= \left(m^2 - \tfrac{1}{2}[\gamma^a, \gamma^b] + \partial_a\partial_b\right)\phi(x) \\ &= (m^2 - \eta^{ab}\partial_a\partial_b)\phi(x) = 0.\end{aligned} \tag{85}$$

The adjoint Majorana field $\overline{\chi}(x) = i\chi^\dagger(x)\gamma^0$ is

$$\overline{\chi}(x) = \overline{\phi}(x)\,(m + \gamma^a \overleftarrow{\partial}_a) \tag{86}$$

in which the derivatives act to the left.

## 5. Dirac field

In this section, we construct a Dirac field from two Majorana fields that describe particles of the same mass.

Suppose there are two spin-one-half particles of the same mass $m$ described by the two operators $a_1(\mathbf{p}, s)$ and $a_2(\mathbf{p}, s)$ which satisfy the anti-commutation relations

$$\left\{a_i(\mathbf{p}, s), a_j^\dagger(\mathbf{p}', s')\right\} = \delta_{ij}\,\delta_{ss'}\delta^3(\mathbf{p} - \mathbf{p}'). \tag{87}$$

Then we have two Majorana fields

$$\chi_{bi}(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \sum_s \left[u_b(\mathbf{p},s)\, a_i(\mathbf{p},s)\, e^{ipx} + v_b(\mathbf{p},s)\, a_i^\dagger(\mathbf{p},s)\, e^{-ipx}\right] \tag{88}$$

for $i = 1, 2$ that satisfy the same Dirac equation

$$(\gamma^a\partial_a + m)\chi_i(x) = 0. \tag{89}$$

So it makes sense to combine them into one Dirac field

$$\psi(x) = \tfrac{1}{\sqrt{2}}[\chi_1(x) + i\chi_2(x)] \tag{90}$$

which satisfies the Dirac equation

$$(\gamma^a\partial_a + m)\,\psi(x) = 0 \tag{91}$$

because its Majorana parts do. In terms of the same spinors (33) and (35), the Dirac field is

$$\psi(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \sum_s [u(\mathbf{p}, s)\, a(\mathbf{p}, s)\, e^{ipx} + v(\mathbf{p}, s)\, a^{c\,\dagger}(\mathbf{p}, s)\, e^{-ipx}] \tag{92}$$

with the complex operators

$$a(\mathbf{p}, s) = \frac{1}{\sqrt{2}}[a_1(\mathbf{p}, s) + i a_2(\mathbf{p}, s)] \tag{93}$$

and

$$a^{c\dagger}(\mathbf{p}, s) = \frac{1}{\sqrt{2}}\left[a_1^\dagger(\mathbf{p}, s) + i a_2^\dagger(\mathbf{p}, s)\right], \tag{94}$$

whence

$$a^c(\mathbf{p}, s) = \frac{1}{\sqrt{2}}[a_1(\mathbf{p}, s) - i a_2(\mathbf{p}, s)]. \tag{95}$$

These complex annihilation and creation operators satisfy the anti-commutation relations

$$\{a(\mathbf{p}, s), a(\mathbf{p}', s')\} = 0 \qquad \{a^c(\mathbf{p}, s), a^c(\mathbf{p}', s')\} = 0 \tag{96}$$

and

$$\{a(\mathbf{p}, s), a^c(\mathbf{p}', s')\} = 0 \qquad \{a(\mathbf{p}, s), a^{c\dagger}(\mathbf{p}', s')\} = 0 \tag{97}$$

as well as

$$\{a(\mathbf{p}, s), a^\dagger(\mathbf{p}', s')\} = \delta_{ss'}\, \delta(\mathbf{p} - \mathbf{p}'). \tag{98}$$

and

$$\{a^c(\mathbf{p}, s), a^{c\dagger}(\mathbf{p}', s')\} = \delta_{ss'}\, \delta(\mathbf{p} - \mathbf{p}'). \tag{99}$$

An action density that leads to the Dirac equation is the sum of two Majorana action densities (74) of the same mass $m$

$$\mathcal{L}_D = \mathcal{L}_{M1} + \mathcal{L}_{M2} = -\tfrac{1}{2}\bar{\psi}\gamma^a\partial_a\psi + \tfrac{1}{2}(\partial_a\bar{\psi})\gamma^a\psi - m\bar{\psi}\psi. \tag{100}$$

### *5.1. Anti-particle field*

Because the Majorana components $\chi_1$ and $\chi_2$ of the Dirac field $\psi$ satisfy the Majorana condition (72), the complex conjugate field $\psi_a^*(x) = \psi_a^\dagger(x)$ multiplied by $\gamma^2$ is the field of the anti-particle

$$\begin{aligned}\psi^c(x) &= \gamma^2\psi^*(x) = \gamma^2\frac{1}{\sqrt{2}}[\chi_1^*(x) - i\chi_2^*(x)] \\ &= \frac{1}{\sqrt{2}}[\chi_1(x) - i\chi_2(x)].\end{aligned} \tag{101}$$

More explicitly, the spinors $u(\mathbf{p}, s)$ and $v(\mathbf{p}, s)$ satisfy the Majorana condition (43) according to which they are interchanged by the operation $*$ followed by $\gamma^2$, and so these operations turn $\psi$ into the charge-conjugate field $\psi_c = \gamma^2\psi^*$

$$\begin{aligned}\psi_c(x) &= \int \frac{d^3p}{(2\pi)^{3/2}} \sum_s [v(\mathbf{p}, s)\, a^\dagger(\mathbf{p}, s)\, e^{-ipx} + u(\mathbf{p}, s)\, a^c(\mathbf{p}, s)\, e^{ipx}] \\ &= \int \frac{d^3p}{(2\pi)^{3/2}} \sum_s [u(\mathbf{p}, s)\, a^c(\mathbf{p}, s)\, e^{ipx} + v(\mathbf{p}, s)\, a^\dagger(\mathbf{p}, s)\, e^{-ipx}]\end{aligned} \tag{102}$$

which is the field of the anti-particle.

### 5.2. Helicity

In the limit of small, positive $m/p_3$, we may infer from the spinor formulas (67)–(70) and from the form (92) of the Dirac field $\psi$ that its upper two components annihilate particles mainly of negative helicity, because of the coefficients $u_1(\mathbf{p}, s)$ and $u_2(\mathbf{p}, s)$, and create anti-particles mainly of positive helicity, because of the coefficients $v_1(\mathbf{p}, s)$ and $v_2(\mathbf{p}, s)$. Similarly, the upper two components of the adjoint field $\psi^\dagger$ create particles mainly of negative helicity, because of the coefficients $u_1^*(\mathbf{p}, s)$ and $u_2^*(\mathbf{p}, s)$, and annihilate anti-particles mainly of positive helicity, because of the coefficients $v_1^*(\mathbf{p}, s)$ and $v_2^*(\mathbf{p}, s)$.

The factor $1 + \gamma_5$ in the charged current of the weak interaction selects the upper two components. Thus in the decay $\mu^- \to \nu_\mu + \mathrm{e}^- + \bar{\nu}_e$ of the muon, the field $\psi_e^\dagger$ creates an electron of mainly negative helicity. The electrons from unpolarized muons mainly have negative helicity. Similarly, in the decay $\mu^+ \to \bar{\nu}_\mu + \mathrm{e}^+ + \nu_e$ of the positive muon, the field $\psi_e$ creates a positron of mainly positive helicity. The $\bar{\nu}_e$ and $\nu_e$ respectively and overwhelmingly have positive and negative helicity.

For the same reason, electrons emitted in beta decay $n \to p + e + \bar{\nu}_e$ tend to be of negative helicity, although the factor $m/|\mathbf{p}_e|$ need not be tiny. The $\bar{\nu}_e$ has positive helicity.

The decay of the charged pion provides another example. The $\pi^-$ can decay into $e + \bar{\nu}_e$ and into $\mu + \bar{\nu}_\mu$. Since the electron is some 200 times lighter than the muon, the available phase space of the $e + \bar{\nu}_e$ channel is 3.49 times greater than that of the $\mu + \bar{\nu}_\mu$ channel. So one would expect that the dominant decay channel would be $\pi^- \to e + \bar{\nu}_e$. Experimentally, however, 99.9877% of the decays go via the channel $\pi^- \to \mu + \bar{\nu}_\mu$. Why? Well, the field $\psi_e^\dagger$ makes an electron of mainly negative helicity, and the field $\nu_e$ creates a $\bar{\nu}_e$ of mainly positive helicity. Now, in the rest frame of the decaying pion, the momenta of the electron and neutrino are (equal and) opposite, and so in a final state composed of the large spinor components, their two parallel spins would add to an angular momentum of $\hbar$. But the pion is a pseudo-scalar meson, and so conservation of angular momentum allows only the small $m_e/(2p_e)$ components of the spinors (67)–(70) to contribute to the amplitude. This effect also slows down the principal decay mode $\pi^- \to \mu + \bar{\nu}_\mu$, but the factor $m_\mu/(2p_\mu)$ is bigger because $m_\mu \approx 207 m_e$ and because $p_e \approx 2.34 p_\mu$. The $e + \bar{\nu}_e$ channel is helicity suppressed relative to the $\mu + \nu_\mu$ channel by the factor $[p_\mu m_e/(p_e m_\mu)]^2 = 4.2 \times 10^{-6}$.

### 5.3. Causality

The Dirac field anti-commutes with itself

$$\{\psi_a(x), \psi_b(y)\} = 0 \tag{103}$$

because of the way it is constructed $\sqrt{2}\psi = \chi_1 + \mathrm{i}\chi_2$ from two fields of the same mass

$$2\{\psi_a(x), \psi_b(y)\} = \{\chi_{1a}(x), \chi_{1b}(y)\} - \{\chi_{2a}(x), \chi_{2b}(y)\} = 0. \tag{104}$$

The vanishing of this anti-commutator also follows from the anti-commutation relations (96)–(99).

Similarly, the anti-commutator of $\psi(x)$ with its adjoint $\overline{\psi}(x) = \mathrm{i}\psi^\dagger(x)\gamma^0$ follows from the anti-commutation relation (78) obeyed by its constituent Majorana fields

$$\begin{aligned}\{\psi_a(x), \overline{\psi}_b(y)\} &= \tfrac{1}{2}\{\chi_{1a}(x), \overline{\chi_{1b}}(y)\} + \tfrac{1}{2}\{\chi_{2a}(x), \overline{\chi_{2b}}(y)\} \\ &= (m - \gamma^c \partial_c)_{ab}\, \Delta(x - y)\end{aligned} \tag{105}$$

in which $\Delta(x-y)$ is the Lorentz-invariant function (76). One also may evaluate this anti-commutator by using the anti-commutation relations (96)–(99) and the spin sums (49) and (52)

$$\{\psi_a(x), \overline{\psi}_b(y)\} = \int \frac{d^3p}{(2\pi)^3} \sum_s (u_a(\mathbf{p}, s)\overline{u}_b(\mathbf{p}, s)\, e^{ip(x-y)} + v_a(\mathbf{p}, s)\overline{v}_b(\mathbf{p}, s)\, e^{-ip(x-y)})$$

$$= \int \frac{d^3p}{(2p^0)(2\pi)^3}((m - i\gamma^c p_c)_{ab}\, e^{ip(x-y)} - (m + i\gamma^c p_c)_{ab}\, e^{-ip(x-y)})$$

$$= (m - \gamma^c \partial_c)_{ab} \Delta(x-y). \tag{106}$$

At equal times, $\Delta(x-y)$ vanishes, and only the time derivative contributes in (106), so

$$\{\psi_a(x), \overline{\psi}_b(y)\}|_{x^0=y^0} = i\gamma^0_{ab}\, \delta(\mathbf{x}-\mathbf{y}) \tag{107}$$

or

$$\left\{\psi_a(x), \psi_b^\dagger(y)\right\}\Big|_{x^0=y^0} = \delta_{ab}\, \delta(\mathbf{x}-\mathbf{y}). \tag{108}$$

### *5.4. Propagator*

Since the Dirac field is the combination (90) of Majorana fields, its propagator $\langle 0|T\{\psi_a(x)\bar{\psi}_b(y)|0\rangle$ is the same as that of a Majorana field (81) of the same mass.

### *5.5. Complex scalar-like field*

The Dirac field also may be obtained from the scalar-like field

$$\Phi(x) = \int \frac{d^3p}{\sqrt{2(2\pi)^3 p^0(p^0+m)}} \sum_{s=-\frac{1}{2}}^{\frac{1}{2}} [u(\mathbf{0}, s)a(\mathbf{p}, s)\, e^{ipx} + v(\mathbf{0}, s)a^{c\dagger}(\mathbf{p}, s)\, e^{-ipx}] \tag{109}$$

as

$$\psi(x) = (m - \gamma^a \partial_a)\Phi(x). \tag{110}$$

The field $\Phi(x)$ is like a scalar field because the static spinors (17) and (18) do not vary with the momentum. It satisfies anti-commutation relations:

$$\{\Phi_a(x), \Phi_b(y)\} = 0. \tag{111}$$

With this construction, the Dirac field satisfies the Dirac equation

$$(\gamma^a \partial_a + m)\psi(x) = 0 \tag{112}$$

because the field $\Phi(x)$ satisfies the Klein–Gordon equation, and it satisfies the anti-commutation relations

$$\{\psi_a(x), \psi_b(y)\} = 0 \tag{113}$$

because the complex scalar-like field $\Phi(x)$ does, equation (111). The adjoint Dirac field $\overline{\psi}(x) = i\psi^\dagger(x)\gamma^0$ is related to the scalar-like field $\Phi(x)$ by

$$\overline{\psi}(x) = \overline{\Phi}(x)\,(m + \gamma^a \overleftarrow{\partial}_a). \tag{114}$$

## 6. Applications to supersymmetry

The relations we have derived for Majorana and Dirac fields are useful in many contexts. As an example of their utility, we shall in this final section use some of them to determine the generators of supersymmetry in the Wess–Zumino model. Such supercharges play key roles in supersymmetric field theories.

### 6.1. Wess–Zumino model

If $\chi$ is a Majorana field, $\chi = \gamma^2\chi^*$, $B$ a real scalar field, and $C$ a real pseudo-scalar field, then the action density of the Wess–Zumino model [5] is

$$\mathcal{L} = -\tfrac{1}{2}\partial_a B\,\partial^a B - \tfrac{1}{2}\partial_a C\,\partial^a C - \tfrac{1}{4}\bar\chi\gamma^a\partial_a\chi + \tfrac{1}{4}(\partial_a\bar\chi)\gamma^a\chi + \tfrac{1}{2}(F^2+G^2)$$
$$+m\left(FB + GC - \tfrac{1}{2}\bar\chi\chi\right) + g[F(B^2-C^2) + 2GBC - \bar\chi(B+\mathrm{i}\gamma_5 C)\chi] \quad (115)$$

in which $(-m/2)\bar\chi\chi$ is a Majorana mass term. The action density $\mathcal{L}$ is said to be supersymmetric because it changes only by a total divergence under the susy transformation

$$\delta B = \bar\chi\alpha \qquad \delta C = -\mathrm{i}\bar\chi\gamma_5\alpha$$
$$\delta\chi = \partial_a(B+\mathrm{i}\gamma_5 C)\gamma^a\alpha + (F-\mathrm{i}\gamma_5 G)\alpha \quad (116)$$
$$\delta F = -\partial_a\bar\chi\,\gamma^a\alpha \qquad \delta G = \mathrm{i}\partial_a\bar\chi\gamma^a\gamma_5\,\alpha$$

in which $\alpha$ is a constant anti-commuting c-number spinor that satisfies the Majorana condition $\alpha = \gamma^2\alpha^*$. The change $\delta\mathcal{L}$ is a total divergence irrespective of whether the fields obey their equations of motion. Some authors write the action density (115) and the susy transformation (116) in terms of $C' = -C$ and $\delta C' = -\delta C$.

We have written the susy transformation (116) exclusively in terms of the spinor $\alpha$ by using the Majorana character of $\chi$ and of $\alpha$ which imply

$$\bar\alpha\chi = \bar\chi\alpha \qquad \text{and} \qquad \bar\alpha\gamma_5\chi = \bar\chi\gamma_5\alpha. \quad (117)$$

Since $\gamma^2(\gamma^a)^* = \gamma^a\gamma^2$, it follows from (116) that the change $\delta\chi$ is also Majorana

$$\delta\chi = \gamma^2\chi^* \quad (118)$$

which in turn implies both

$$\delta\bar\chi\chi = \bar\chi\delta\chi \qquad \text{and} \qquad \delta\bar\chi\gamma_5\chi = \bar\chi\gamma_5\delta\chi \quad (119)$$

and with $\gamma^2(\gamma^a)^\top\gamma^0 = \gamma^0\gamma^a\gamma^2$ also

$$\delta\bar\chi\gamma^a\chi = -\bar\chi\gamma^a\delta\chi. \quad (120)$$

Incidentally, $\overline{\delta\chi} = \delta\bar\chi$.

### 6.2. Change in action density

Under the susy transformation (116), the change in the action density is

$$\delta\mathcal{L} = \big\{-\partial_a B\partial^a\bar\chi + \mathrm{i}\partial_a C\partial^a\bar\chi\gamma_5 - \tfrac{1}{2}\bar\chi\gamma^a\partial_a[\partial_b(B+\mathrm{i}\gamma_5 C)\gamma^b + (F-\mathrm{i}\gamma_5 G)]$$
$$+\tfrac{1}{2}\partial_a\bar\chi\gamma^a[\partial_b(B+\mathrm{i}\gamma_5 C)\gamma^b + (F-\mathrm{i}\gamma_5 G)] - F\partial_a\bar\chi\gamma^a + \mathrm{i}G\partial_a\bar\chi\gamma^a\gamma_5$$
$$+m\{F\bar\chi - B\partial_a\bar\chi\gamma^a - \mathrm{i}G\bar\chi\gamma_5 + \mathrm{i}C\partial_a\bar\chi\gamma^a\gamma_5$$
$$-\bar\chi[\partial_a(B+\mathrm{i}\gamma_5 C)\gamma^a + (F-\mathrm{i}\gamma_5 G)]\} - gB^2\partial_a\bar\chi\gamma^a + gC^2\partial_a\bar\chi\gamma^a$$
$$+2gFB\bar\chi + 2\mathrm{i}gFC\bar\chi\gamma_5 + 2\mathrm{i}gBC\partial_a\bar\chi\gamma^a\gamma_5 - 2\mathrm{i}GB\bar\chi\gamma_5 + 2gGC\bar\chi$$
$$-2g\bar\chi B[\partial_a(B+\mathrm{i}\gamma_5 C)\gamma^a + F - \mathrm{i}\gamma_5 G]$$
$$-2\mathrm{i}g\bar\chi\gamma_5 C[\partial_a(B+\mathrm{i}\gamma_5 C)\gamma^a + F - \mathrm{i}\gamma_5 G] - g\bar\chi\chi\bar\chi - g\bar\chi\gamma_5\chi\bar\chi\gamma_5\big\}\alpha. \quad (121)$$

After several cancellations, $\delta\mathcal{L}$ simplifies to

$$\delta\mathcal{L} = \big\{-\partial_a B\partial^a\bar\chi + \mathrm{i}\partial_a C\partial^a\bar\chi\gamma_5 - \tfrac{1}{2}\bar\chi\gamma^a\gamma^b\partial_a\partial_b(B-\mathrm{i}\gamma_5 C) + \tfrac{1}{2}\partial_a\bar\chi\gamma^a\gamma^b\partial_b(B-\mathrm{i}\gamma_5 C)$$
$$-\tfrac{1}{2}\partial_a(F\bar\chi\gamma^a) + \tfrac{1}{2}\mathrm{i}\partial_a(G\bar\chi\gamma^a\gamma_5) + m[-\partial_a(B\bar\chi\gamma^a) + \mathrm{i}\partial_a(C\bar\chi\gamma^a\gamma_5)]$$
$$-g[\partial_a(\bar\chi(B+\mathrm{i}\gamma_5 C)^2\gamma^a) + \bar\chi\chi\bar\chi + \bar\chi\gamma_5\chi\bar\chi\gamma_5]\big\}\alpha. \quad (122)$$

The terms that are cubic in $\chi$ cancel. To see this, we note that the Majorana condition (72) $\chi = \gamma^2\chi^*$ and $(\gamma^2)^2 = 1$ imply that $\chi^* = \gamma^2\chi$ and so that $\chi^\dagger = \chi^\top(\gamma^2)^\top = \chi^\top\gamma^2$, whence $\bar\chi = \mathrm{i}\chi^\dagger\gamma^0 = \mathrm{i}\chi^\top\gamma^2\gamma^0$. It follows that

$$-\bar\chi\chi\bar\chi\alpha = \chi^\top\begin{pmatrix}\sigma_2 & 0\\ 0 & -\sigma_2\end{pmatrix}\chi\chi^\top\begin{pmatrix}\sigma_2 & 0\\ 0 & -\sigma_2\end{pmatrix}\alpha \tag{123}$$

and

$$-\bar\chi\gamma_5\chi\bar\chi\gamma_5\alpha = \chi^\top\begin{pmatrix}\sigma_2 & 0\\ 0 & -\sigma_2\end{pmatrix}\chi\chi^\top\begin{pmatrix}\sigma_2 & 0\\ 0 & \sigma_2\end{pmatrix}\alpha. \tag{124}$$

The sum of these two terms contains only products $\chi_i\chi_j\chi_k$ in which all the indices are either 1 or 2 or all are 3 or 4; all products with three different indices, like $\chi_1\chi_2\chi_3$, cancel. So every surviving term contains a product of two identical fields $\chi_i(x)\chi_i(x)$ for some $i$. But the equal-time anti-commutation relation (76) (to wit, $\{\chi_a(x), \chi_b(y)\} = \gamma^2_{ab}\delta(\mathbf{x}-\mathbf{y})$) implies that such terms vanish. So the terms with three $\chi$'s cancel.

Next, the anti-commutation relations (9) of the gamma matrices imply that

$$\gamma^a\gamma^b\partial_a\partial_b = \gamma^b\gamma^a\partial_a\partial_b = (-\gamma^a\gamma^b + 2\eta^{ab})\partial_a\partial_b \tag{125}$$

so we can write $\delta\mathcal{L}$ as the total divergence

$$\delta\mathcal{L} = \partial_a K^a \tag{126}$$

of the current

$$K^a = \big[-\bar\chi\partial^a(B + \mathrm{i}\gamma_5 C) + \tfrac{1}{2}\bar\chi\gamma^a\gamma^b\partial_b(B - \mathrm{i}\gamma_5 C) - \tfrac{1}{2}\bar\chi\gamma^a(F - \mathrm{i}\gamma_5 G) \\ - m\bar\chi\gamma^a(B - \mathrm{i}\gamma_5 C) - g\bar\chi(B + \mathrm{i}\gamma_5 C)^2\gamma^a\big]\alpha. \tag{127}$$

Thus whether or not the fields satisfy their equations of motion, the change in the Wess–Zumino action density is a total divergence, and in this sense, the Wess–Zumino action is supersymmetric.

### *6.3. Noether current*

For a general action density $\mathcal{L}(\phi)$, Lagrange's equations

$$\partial_a\frac{\partial\mathcal{L}(\phi)}{\partial\partial_a\phi_i} = \frac{\partial\mathcal{L}(\phi)}{\partial\phi_i} \tag{128}$$

and the identity $\delta\partial_a\phi_i = \partial_a\delta\phi_i$ imply that any first-order change

$$\delta\mathcal{L}(\phi) = \frac{\partial\mathcal{L}(\phi)}{\partial\partial_a\phi_i}\delta\partial_a\phi_i + \frac{\partial\mathcal{L}(\phi)}{\partial\phi_i}\delta\phi_i \tag{129}$$

is a total divergence

$$\delta\mathcal{L}(\phi) = \partial_a\left(\frac{\partial\mathcal{L}(\phi)}{\partial\partial_a\phi_i}\delta\phi_i\right) = \partial_a J^a \tag{130}$$

of a Noether current

$$J^a = \frac{\partial\mathcal{L}(\phi)}{\partial\partial_a\phi_i}\delta\phi_i. \tag{131}$$

The Noether current $J^a$ is conserved $\partial_a J^a = 0$ by the equations of motion (128), if the action density is invariant $\delta\mathcal{L} = 0$ to first order. The Noether current of the supersymmetry transformation (116) is not conserved.

### 6.4. Wess–Zumino Noether current

The change in the Wess–Zumino action density (115) is by (120), (130) and (131) the divergence

$$\delta\mathcal{L} = \partial_a J^a \tag{132}$$

of the susy Noether current

$$\begin{aligned} J^a &= \tfrac{1}{4}\delta\bar{\chi}\gamma^a\chi - \tfrac{1}{4}\bar{\chi}\gamma^a\delta\chi - (\partial^a B)\delta B - (\partial^a C)\delta C \\ &= -\tfrac{1}{2}\bar{\chi}\gamma^a\delta\chi - (\partial^a B)\delta B - (\partial^a C)\delta C \end{aligned} \tag{133}$$

or

$$J^a = \left[-\tfrac{1}{2}\bar{\chi}\gamma^a\gamma^b\partial_b(B - \mathrm{i}\gamma_5 C) - \tfrac{1}{2}\bar{\chi}\gamma^a(F - \mathrm{i}\gamma_5 G) - \bar{\chi}\partial^a(B - \mathrm{i}\gamma_5 C)\right]\alpha. \tag{134}$$

The current $J^a$ is Hermitian $J_a^\dagger = J_a$ as it should be, but because the change $\delta\mathcal{L}$ is a non-zero total divergence, it is not conserved.

### 6.5. The susy current

Although neither the current $J^a$ nor the current $K^a$ is conserved, by (126) and (130), the divergence of each of them is the change $\delta\mathcal{L}$ in the action density

$$\partial_a K^a = \partial_a J^a = \delta\mathcal{L}. \tag{135}$$

So the difference of the two currents

$$\begin{aligned} S^a &= K^a - J^a \\ &= \bar{\chi}\gamma^a[\gamma^b\partial_b - m - g(B - \mathrm{i}\gamma_5 C)](B - \mathrm{i}\gamma_5 C)\alpha \end{aligned} \tag{136}$$

has zero divergence

$$\partial_a S^a = 0 \tag{137}$$

and is conserved. This current $S^a$ is the conserved susy current of the Wess–Zumino action. It contains no auxiliary fields.

### 6.6. Supercharges

The supercharge $\bar{Q}$ multiplied by the spinor $\alpha$ is the spatial integral of $S^0$

$$\begin{aligned} \bar{Q}\alpha &= \int \mathrm{d}^3x\, S^0 \\ &= \int \mathrm{d}^3x\, \bar{\chi}\gamma^0[\gamma^b\partial_b - m - g(B - \mathrm{i}\gamma_5 C)](B - \mathrm{i}\gamma_5 C)\alpha \\ &= -\mathrm{i}\int \mathrm{d}^3x\, \chi^\dagger[\gamma^b\partial_b - m - g(B - \mathrm{i}\gamma_5 C)](B - \mathrm{i}\gamma_5 C)\alpha. \end{aligned} \tag{138}$$

By inserting unity in the form $I = (\gamma^2)^2$, one may show that the supercharge satisfies the Majorana condition

$$Q = \gamma^2 Q^* \tag{139}$$

so that

$$\begin{aligned} \bar{Q}\alpha &= \bar{\alpha}Q \\ &= \bar{\alpha}\int \mathrm{d}^3x\{\gamma^a\partial_a(B + \mathrm{i}\gamma_5 C) + [m + g(B - \mathrm{i}\gamma_5 C)](B - \mathrm{i}\gamma_5 C)\}\gamma^0\chi. \end{aligned} \tag{140}$$

The susy transformation rules (116) may be written as

$$\mathrm{i}\delta\mathcal{O}(x) = [\bar{Q}\alpha, \mathcal{O}(x)] = [\bar{\alpha}Q, \mathcal{O}(x)]. \tag{141}$$

The change in the field $B(x)$ is

$$\begin{aligned}\mathrm{i}\delta B(x) &= [\bar{Q}\alpha, B(x)] \\ &= -\mathrm{i}\int \mathrm{d}^3x'\, \chi^\dagger\gamma^0[\partial_0 B(x'), B(x)]\alpha\end{aligned} \tag{142}$$

which the equal-time commutation relation $[B(x), \partial_0 B(x')] = \mathrm{i}\delta(\mathbf{x}-\mathbf{x}')$ reduces to

$$\mathrm{i}\delta B(x) = -\mathrm{i}\int \mathrm{d}^3x'\, \chi^\dagger\gamma^0(-\mathrm{i})\delta(\mathbf{x}-\mathbf{x}') = \mathrm{i}\bar{\chi}\alpha \tag{143}$$

in agreement with (116). Similarly, the change in $C(x)$ is

$$\begin{aligned}\mathrm{i}\delta C(x) &= [\bar{Q}\alpha, C(x)] \\ &= -\mathrm{i}\int \mathrm{d}^3x'\, \chi^\dagger\gamma^0(-\mathrm{i}\gamma_5)[\partial_0 C(x'), C(x)]\alpha = \bar{\chi}\gamma_5\alpha\end{aligned} \tag{144}$$

as in (116).

By (141), the change in $\chi$ is

$$\begin{aligned}\mathrm{i}\delta\chi_a(x) &= [\bar{Q}\alpha, \chi_a(x)] \\ &= \mathrm{i}\int \mathrm{d}^3x'\, \{\chi^\dagger(x'), \chi_a(x)\}[\gamma^b\partial_b - m - g(B-\mathrm{i}\gamma_5 C)](B-\mathrm{i}\gamma_5 C)\alpha\end{aligned} \tag{145}$$

and so since $\left\{\chi_b^\dagger(x'), \chi_a(x)\right\} = \delta_{ab}\delta(\mathbf{x}-\mathbf{x}')$ (the equal-time anti-commutation relation (80)), $\delta\chi$ is

$$\delta\chi = [\gamma^b\partial_b - m - g(B-\mathrm{i}\gamma_5 C)](B-\mathrm{i}\gamma_5 C)\alpha. \tag{146}$$

The auxiliary fields $F$ and $G$ occur quadratically and without their derivatives in the action density (115); their field equations are

$$F = -mB - g(B^2 - C^2) \tag{147}$$

and

$$G = -mC - 2gBC. \tag{148}$$

In terms of them, the change in $\chi$ is

$$\delta\chi = \partial_a(B + \mathrm{i}\gamma_5 C)\gamma^a\alpha + (F - \mathrm{i}\gamma_5 G)\alpha \tag{149}$$

as in (116).

The supercharges (138) and (141) obey the anti-commutation relation

$$\{Q_a, \bar{Q}_b\} = -2\mathrm{i}P_c\gamma^c_{ab} \tag{150}$$

which is a fundamental property of the algebra of supersymmetric theories with a single Majorana supercharge.

The supercharges $Q_f$ of the free theory are given by (138) or (141) with $g = 0$. Because the spinors $u(\mathbf{p}, s)$ and $v(\mathbf{p}, s)$ satisfy the Dirac equation in momentum space (39) and (40), one may write $Q_f$ as

$$\begin{aligned}Q_f = \mathrm{i}\int \mathrm{d}^3p\sqrt{2p^0}\sum_{s=-\frac{1}{2}}^{\frac{1}{2}} &[(b(p) - \mathrm{i}\gamma_5 c(p))v(\mathbf{p}, s)a^\dagger(\mathbf{p}, s) \\ &- (b^\dagger(p) - \mathrm{i}\gamma_5 c^\dagger(p))u(\mathbf{p}, s)a(\mathbf{p}, s)]\end{aligned} \tag{151}$$

from which it is clear that they annihilate the vacuum of the free theory

$$Q_{fa}|0\rangle = 0 \tag{152}$$

as they must since supersymmetry is unbroken and the energy of the ground state $|0\rangle$ is zero in the free theory.

## Acknowledgment

Thanks to Randolph Reeder for helpful conversations.

## References

[1] Weinberg S 1964 Feynman rules for any spin *Phys. Rev.* **133** B1318–32
[2] Weinberg S 1964 Feynman rules for any spin: II. Massless particles *Phys. Rev.* **134** B882–96
[3] Weinberg S 1969 Feynman rules for any spin: III *Phys. Rev.* **181** 1893–99
[4] Weinberg S 1995 *The Quantum Theory of Fields, vol I: Foundations* (Cambridge: Cambridge University Press)
[5] Wess J and Zumino B 1974 Supergauge transformations in four dimensions *Nucl. Phys.* B **70** 39–50