6.29 Derive the Yukawa potential (6.392) as the Green's function for the modified Helmholtz equation (6.391).

6.30 Derive the relation $\rho = \overline{\rho}(\overline{a}/a)^{3(1+w)}$ between the energy density ρ and the Robertson-Walker scale factor $a(t)$ from the conservation law $d\rho/da = -3(\rho + p)/a$ and the equation of state $p = w\rho$.

6.31 For a closed universe ($k = 1$) of radiation ($w = 1/3$), use Friedmann's equations (6.418 & 6.419) to derive the solution (11.448) subject to the boundary condition $a(0) = 0$. When does the universe collapse in a big crunch?

6.32 For a flat universe ($k = 0$) of matter ($w = 0$), use Friedmann's equations (6.418 & 6.419) to derive the solution (11.454) subject to the boundary condition $a(0) = 0$.

6.33 Derive the time evolution of $a(t)$ for a flat ($k = 0$) universe dominated by radiation ($w = 1/3$) subject to the boundary condition $a(0) = 0$. Use (6.419).

6.34 Derive the time evolution of $a(t)$ for an open ($k = -1$) universe with only dark energy ($w = -1$) subject to the boundary condition $a(0) = 0$. Use (6.419).

6.35 Use Friedmann's equations (6.418 & 6.419) to derive the evolution of $a(t)$ for a closed universe dominated by dark energy subject to the boundary condition $a(0) = \sqrt{3/8\pi G\rho}$ in which ρ is a constant density of dark energy.

6.36 Use Friedmann's equations (6.418 & 6.419) to derive the evolution of $a(t)$ for a flat ($k = 0$) expanding universe dominated by dark energy ($w = -1$) subject to the boundary condition $a(0) = \alpha$ in which ρ is a constant density of dark energy.

6.37 Derive the soliton solution (6.432) from the energy equation (6.431).

6.38 Find the solution of the differential equation $-f''(x) - f(x) = 1$ that satisfies the boundary conditions $f(-\pi) = 0 = f'(\pi)$. Hint: Use example 6.45.

6.39 Show that for any constants c_k, the sum of the exponentials

$$f(x) = c_1 e^{z_1 x} + c_2 e^{z_2 x} + \cdots + c_n e^{z_n x}$$

in which the z_k's are the n roots of the algebraic equation

$$0 = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

is a solution of the homogeneous ordinary differential equation

$$0 = a_0 f(x) + a_1 f'(x) + a_2 f''(x) + \cdots + a_n f^{(n)}(x)$$

with constant coefficients a_k. When the roots are all different, $f(x)$ is the most general solution.

6.40 Find two linearly independent solutions of the ODE $f'' - 2f' + f = 0$.