5.43 (a) Perform the z-integral in Eq.(5.340). (b) Use the result of the part (a) to find the commutator $[L_m, L_n]$ of the **Virasoro algebra**. Hint: use the Laurent series (5.336).

5.44 Assume that $\epsilon(z)$ is analytic in a disk that contains a tiny circular contour C_w about the point w as in Fig 5.10. Do the contour integral

$$\oint_{C_w} \epsilon(z) \left[\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{T'(w)}{z-w} \right] \frac{dz}{2\pi i} \tag{5.356}$$

and express your result in terms of $\epsilon(w)$, $T(w)$, and their derivatives.

5.45 Show that if the coefficients a_k of the equation $0 = a_0 + a_1 z + \cdots + a_n z^n$ are real, then its n roots z_k are real or come in pairs that are complex conjugates, z_ℓ and z_ℓ^*, of each other.

5.46 Show that if a is not an integer, then the sum of the tiny ccw integrals about the points $z = n$ of example 5.38 is

$$\sum_{n=-\infty}^{\infty} \oint_n \frac{\csc \pi z}{(z-a)^2}\, dz = 2i \sum_{n=-\infty}^{\infty} (-1)^n \frac{1}{(n-a)^2}.$$

5.47 Use the trick of example 5.38 with $\csc \pi z \to \cot \pi z$ to show that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n-a)^2} = \frac{\pi^2}{\sin^2 \pi a}$$

as long as a is not an integer.