$2n \times 2n$ antihermitian matrix but also an $n \times n$ matrix of quaternions. Thus the matrices

$$D(\alpha) = e^{i\alpha_a t_a} \tag{10.191}$$

are both unitary $2n \times 2n$ matrices and $n \times n$ quaternionic matrices and so are elements of the group $Sp(2n)$.

Example 10.26 ($Sp(2) = SU(2)$) There is no 1×1 anti-symmetric matrix, and there is only one 1×1 symmetric matrix. So the generators t_a of the group $Sp(2)$ are the Pauli matrices $t_a = \sigma_a$, and $Sp(2) = SU(2)$. The elements $g(\alpha)$ of $SU(2)$ are quaternions of unit norm (exercise 10.20), and so the product $g(\alpha)q$ is a quaternion

$$\|g(\alpha)q\|^2 = \det(g(\alpha)q) = \det(g(\alpha)) \det q = \det q = \|q\|^2 \tag{10.192}$$

with the same squared norm. □

Example 10.27 ($SO(4) = SU(2) \otimes SU(2)$) If g and h are any two elements of the group $SU(2)$, then the squared norm (10.182) of the quaternion $q(x) = x_0 + i\boldsymbol{\sigma} \cdot \boldsymbol{x}$ is invariant under the transformation $q(x') = g\, q(x)\, h^{-1}$, that is, $x_0'^2 + x_1'^2 + x_2'^2 + x_3'^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2$. So $x \to x'$ is an $SO(4)$ rotation of the 4-vector x. The Lie algebra of $SO(4)$ thus contains two commuting invariant $SU(2)$ subalgebras and so is semisimple. □

Example 10.28 ($Sp(4) = SO(5)$) Apart from scale factors, there are three real symmetric 2×2 matrices $S_1 = \sigma_1$, $S_2 = I$, and $S_3 = \sigma_3$ and one imaginary anti-symmetric 2×2 matrix $A = \sigma_2$. So there are 10 generators of $Sp(4) = SO(5)$

$$\begin{aligned} t_1 &= I \otimes \sigma_2 = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}, \quad t_{k1} = \sigma_k \otimes \sigma_1 = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \\ t_{k2} &= \sigma_k \otimes I = \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}, \quad t_{k3} = \sigma_k \otimes \sigma_3 = \begin{pmatrix} \sigma_k & 0 \\ 0 & -\sigma_k \end{pmatrix} \end{aligned} \tag{10.193}$$

where k runs from 1 to 3. □

We may see $Sp(2n)$ from a different viewpoint if we use (10.179) to write the quadratic form $\|q\|^2$ in terms of a $2n \times 2n$ matrix J that has n copies of