$(2j+1) \times (2j+1)$ identity matrix

$$\left(J_a^{(j)}\right)^2 = j(j+1)\, I. \tag{10.105}$$

Combinations of generators that are a multiple of the identity are called **Casimir operators**.

Example 10.18 (Spin 2) For $j = 2$, the spin-two matrices $J_+^{(2)}$ and $J_3^{(2)}$ are

$$J_+^{(2)} = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad J_3^{(2)} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix} \tag{10.106}$$

and $J_- = \left(J_+^{(2)}\right)^\dagger$. □

The tensor product of any two irreducible representations $D^{(j)}$ and $D^{(k)}$ of $SU(2)$ is equivalent to the direct sum of all the irreducible representations D^ℓ for $|j-k| \le \ell \le j+k$

$$D^{(j)} \otimes D^{(k)} = \bigoplus_{\ell=|j-k|}^{j+k} D^\ell \tag{10.107}$$

each D^ℓ occurring once.

Example 10.19 (Addition theorem) The spherical harmonics $Y_{\ell m}(\theta,\phi) = \langle \hat{\boldsymbol{n}}|\ell, m\rangle$ of section 8.13 transform under the $(2\ell+1)$-dimensional representation D^ℓ of the rotation group. If a rotation R takes the unit vector $\hat{\boldsymbol{n}}$ into the unit vector $\hat{\boldsymbol{n}}'$, so that $|\hat{\boldsymbol{n}}'\rangle = U(R)|\hat{\boldsymbol{n}}\rangle$, then summing over m' from $-\ell$ to ℓ, we get

$$\begin{aligned} Y^*_{\ell,m}(\theta',\phi') &= \langle \ell, m|\hat{\boldsymbol{n}}'\rangle = \langle \ell, m|U(R)|\hat{\boldsymbol{n}}\rangle \\ &= \langle \ell, m|U(R)|\ell, m'\rangle\langle \ell, m'|\hat{\boldsymbol{n}}\rangle = D^\ell(R)_{m,m'} Y^*_{\ell,m'}(\theta,\phi). \end{aligned}$$

Suppose now that a rotation R maps $|\hat{\boldsymbol{n}}_1\rangle$ and $|\hat{\boldsymbol{n}}_2\rangle$ into $|\hat{\boldsymbol{n}}'_1\rangle = U(R)\,|\hat{\boldsymbol{n}}_1\rangle$ and $|\hat{\boldsymbol{n}}'_2\rangle = U(R)\,|\hat{\boldsymbol{n}}_2\rangle$. Then summing over the repeated indices m, m', and m'' from $-\ell$ to ℓ, we find

$$Y_{\ell,m}(\theta'_1,\phi'_1) Y^*_{\ell,m}(\theta'_2,\phi'_2) = D^\ell(R)^*_{m,m'} Y_{\ell,m'}(\theta_1,\phi_1) D^\ell(R)_{m,m''} Y^*_{\ell,m''}(\theta_2,\phi_2).$$

In this equation, the matrix element $D^\ell(R)^*_{m,m'}$ is

$$D^\ell(R)^*_{m,m'} = \langle \ell, m|U(R)|\ell, m'\rangle^* = \langle \ell, m'|U^\dagger(R)|\ell, m\rangle = D^\ell(R^{-1})_{m',m}.$$

Thus, since D^ℓ is a representation of the rotation group, the product of the two D^ℓ's in (10.19) is

$$\begin{aligned} D^\ell(R)^*_{m,m'} D^\ell(R)_{m,m''} &= D^\ell(R^{-1})_{m',m} D^\ell(R)_{m,m''} \\ &= D^\ell(R^{-1}R)_{m',m''} = D^\ell(I)_{m',m''} = \delta_{m',m''}. \end{aligned}$$

Thus, as long as the same rotation R maps $\hat{\boldsymbol{n}}_1$ into $\hat{\boldsymbol{n}}'_1$ and $\hat{\boldsymbol{n}}_2$ into $\hat{\boldsymbol{n}}'_2$, then we have

$$\sum_{m=-\ell}^{\ell} Y_{\ell,m}(\theta'_1, \phi'_1) Y^*_{\ell,m}(\theta'_2, \phi'_2) = \sum_{m=-\ell}^{\ell} Y_{\ell,m}(\theta_1, \phi_1) Y^*_{\ell,m}(\theta_2, \phi_2).$$

We choose the rotation R as the product of a rotation that maps $\hat{\boldsymbol{n}}_2$ into $\hat{\boldsymbol{z}} = (0, 0, 1)$ and a rotation about the z axis that maps $\hat{\boldsymbol{n}}_1$ into the x-z plane. We then have $Y^*_{\ell,m}(\theta'_2, \phi'_2) = Y^*_{\ell,m}(0, 0)$ and $Y_{\ell,m}(\theta'_1, \phi'_1) = Y_{\ell,m}(\theta, 0)$ in which θ is the angle between the unit vectors $\hat{\boldsymbol{n}}_1$ and $\hat{\boldsymbol{n}}_2$, $\cos\theta = \hat{\boldsymbol{n}}_1 \cdot \hat{\boldsymbol{n}}_2$. The vanishing (8.106) at $\theta = 0$ of the associated Legendre functions $P_{\ell,m}$ for $m \neq 0$ and the definitions (8.4, 8.99, & 8.110–8.112) say that $Y^*_{\ell,m}(0, 0) = \sqrt{(2\ell + 1)/4\pi}\, \delta_{m,0}$, and that $Y_{\ell,0}(\theta, 0) = \sqrt{(2\ell + 1)/4\pi}\, P_\ell(\cos\theta)$. Thus, our identity (10.19) gives us the **addition theorem** (8.121)

$$P_\ell(\hat{\boldsymbol{n}}_1 \cdot \hat{\boldsymbol{n}}_2) = \frac{2\ell + 1}{4\pi} \sum_{m=-\ell}^{\ell} Y_{\ell,m}(\theta_1, \phi_1) Y^*_{\ell,m}(\theta_2, \phi_2)$$

for the spherical harmonics. □

Under a rotation R, a field $\psi_\ell(x)$ that transforms under the $D^{(j)}$ representation of $SU(2)$ responds as

$$U(R)\, \psi_\ell(x)\, U^{-1}(R) = D^{(j)}_{\ell\ell'}(R^{-1})\, \psi_{\ell'}(Rx). \tag{10.108}$$

Example 10.20 (Spin and Statistics) Suppose $|a, m\rangle$ and $|b, m\rangle$ are any eigenstates of the rotation operator J_3 with eigenvalue m (in units with $\hbar = c = 1$). Let u and v be any two points whose separation $u - v$ is space-like $(u - v)^2 > 0$. Then in some Lorentz frame, the two points are at the same time t, and we may chose our coordinate system so that $u' = (t, x, 0, 0)$ and $v' = (t, -x, 0, 0)$. Let U be the unitary operator that represents a right-handed rotation by π about the 3-axis or z-axis of this Lorentz frame. Then

$$U|a, m\rangle = e^{-im\pi}|a, m\rangle \quad \text{and} \quad \langle b, m|U^{-1} = \langle b, m|e^{im\pi}. \tag{10.109}$$