in which C is a constant of integration. □

Example 6.48 (Principle of least action) If $\phi(x)$ is a scalar field, and $L(\phi)$ is its action density, then its action $S[\phi]$ is the integral over all of space-time

$$S[\phi] = \int L(\phi(x))\, d^4x.$$

The principle of least (or stationary) action says that the fields $\phi(x)$ that satisfy the classical equations of motion are those for which the first-order change in the action due to any tiny variation $\delta\phi(x)$ vanishes, $\delta S[\phi] = 0$. It will be sufficient to consider variations $\delta\phi(x)$ that vanish as for infinitely large values of any of the coordinates x^a for $a = 0 \ldots 3$. And to keep things simple, we'll assume that the action (or Lagrange) density $L(\phi)$ is a function only of the field ϕ and its first derivatives $\partial_a \phi = \partial\phi/\partial x^a$. The first-order change in the action then is

$$\delta S[\phi] = \int \left[\frac{\partial L}{\partial \phi} \delta\phi + \frac{\partial L}{\partial(\partial_a \phi)} \delta(\partial_a \phi) \right] d^4x.$$

Now $\delta(\partial_a\phi) = \partial_a(\phi + \delta\phi) - \partial_a\phi = \partial_a \delta\phi$. And so we may integrate by parts and drop the surface terms

$$\begin{aligned} \delta S[\phi] &= \int \left[\frac{\partial L}{\partial \phi} \delta\phi + \frac{\partial L}{\partial(\partial_a \phi)} \partial_a(\delta\phi) \right] d^4x \\ &= \int \left[\frac{\partial L}{\partial \phi} - \partial_a \frac{\partial L}{\partial(\partial_a \phi)} \right] \delta\phi \, d^4x \end{aligned}$$

because $\delta\phi$ vanishes on the surface at infinity. This first-order variation is zero, $\delta S[\phi] = 0$, for $\delta\phi$ that are tiny but arbitrary at finite space-time points x only if the field $\phi(x)$ satisfies Lagrange's equations

$$\partial_a \frac{\partial L}{\partial(\partial_a \phi)} \equiv \frac{\partial}{\partial x^a} \left[\frac{\partial L}{\partial(\partial\phi/\partial x^a)} \right] = \frac{\partial L}{\partial \phi}$$

which are the classical equations of motion. □

The equations of particle physics are nonlinear. Physicists usually use perturbation theory to cope with the nonlinearities. But occasionally they focus on the nonlinearities and treat the fields classically or semi-classically. To keep things relatively simple, we'll work in a space-time of only two