$T(z)$ are defined by its Laurent series

$$T(z) = \sum_{n=-\infty}^{\infty} \frac{L_n}{z^{n+2}} \tag{5.336}$$

and the inverse relation

$$L_n = \frac{1}{2\pi i} \oint z^{n+1}\, T(z)\, dz. \tag{5.337}$$

Thus the commutator of two modes involves two loop integrals

$$[L_m, L_n] = \left[\frac{1}{2\pi i} \oint z^{m+1}\, T(z)\, dz, \frac{1}{2\pi i} \oint w^{n+1}\, T(w)\, dw \right] \tag{5.338}$$

which we may deform as long as we cross no poles. Let's hold w fixed and deform the z loop so as to keep the T's radially ordered when z is near w as in Fig. 5.10. The operator-product expansion of the radially ordered product $\mathcal{R}\{T(z)T(w)\}$ is

$$\mathcal{R}\{T(z)T(w)\} = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2} T(w) + \frac{1}{z-w} T'(w) + \dots \tag{5.339}$$

in which the prime means derivative, c is a constant, and the dots denote terms that are analytic in z and w. The commutator introduces a minus sign that cancels most of the two contour integrals and converts what remains into an integral along a tiny circle C_w about the point w as in Fig. 5.10

$$[L_m, L_n] = \oint \frac{dw}{2\pi i} w^{n+1} \oint_{C_w} \frac{dz}{2\pi i} z^{m+1} \left[\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{T'(w)}{z-w} \right]. \tag{5.340}$$

After doing the z-integral, which is left as a homework exercise (5.43), one may use the Laurent series (5.336) for $T(w)$ to do the w-integral, which one may choose to be along a tiny circle about $w = 0$, and so find the commutator

$$[L_m, L_n] = (m-n)\, L_{m+n} + \frac{c}{12}\, m(m^2-1)\, \delta_{m+n,0} \tag{5.341}$$

of the Virasoro algebra.

Example 5.38 (Using ghost contours to sum series) Consider the integral

$$I = \oint_C \frac{\csc \pi z}{(z-a)^2}\, dz$$

along the ccw rectangular contour C from $z = N + 1/2 - iY$ to $z = N + 1/2 + iY$ to $z = -N - 1/2 + iY$ to $z = -N - 1/2 - iY$ and back to

$z = N + 1/2 - iY$ in which N is a positive integer, and a is not an integer. In the twin limits $N \to \infty$ and $Y \to \infty$, the integral vanishes because on the contour $1/|z-a|^2 \approx 1/N^2$ or $1/Y^2$ while $|\csc \pi z| \le 1$. We now shrink the contour down to tiny circles about the poles of $\csc \pi z$ at all the integers, $z = n$ and about the nonintegral value, $z = a$. By Cauchy's integral formula (5.34), the tiny contour integral around $z = a$ is

$$\oint_a \frac{\csc \pi z}{(z-a)^2} \, dz = 2\pi i \frac{d \csc \pi z}{dz}\bigg|_{z=a} = -\, 2\pi^2 i \, \frac{\cos \pi a}{\sin^2 \pi a}.$$

In the twin limits $N \to \infty$ and $Y \to \infty$, the tiny ccw integrals around the poles of $1/\sin \pi z$ at $z = n\pi$ are (exercise 5.46)

$$\sum_{n=-\infty}^{\infty} \oint_n \frac{\csc \pi z}{(z-a)^2} \, dz = 2i \sum_{n=-\infty}^{\infty} (-1)^n \frac{1}{(n-a)^2}.$$

We thus have the sum rule

$$\sum_{n=-\infty}^{\infty} (-1)^n \frac{1}{(n-a)^2} = \pi^2 \cot \pi a \, \csc \pi a.$$

□

Exercises

5.1 Compute the two limits (5.6) and (5.7) of example 5.2 but for the function $f(x, y) = x^2 - y^2 + 2ixy$. Do the limits now agree? Explain.

5.2 Show that if $f(z)$ is analytic in a disk, then the integral of $f(z)$ around a tiny (isosceles) triangle of side $\epsilon \ll 1$ inside the disk is zero to order ϵ^2.

5.3 Derive the two integral representations (5.46) for Bessel's functions $J_n(t)$ of the first kind from the integral formula (5.45). Hint: Think of the integral (5.45) as running from $-\pi$ to π.

5.4 Do the integral

$$\oint_C \frac{dz}{z^2 - 1}$$

in which the contour C is counter-clockwise about the circle $|z| = 2$.

5.5 The function $f(z) = 1/z$ is analytic in the region $|z| > 0$. Compute the integral of $f(z)$ counter-clockwise along the unit circle $z = e^{i\theta}$ for $0 \le \theta \le 2\pi$. The contour lies entirely within the domain of analyticity of the function $f(z)$. Did you get zero? Why? If not, why not?