Its derivatives, which we'll call $\mathcal{P}^\tau_\mu$ and $\mathcal{P}^\sigma_\mu$, are

$$\mathcal{P}^\tau_\mu = \frac{\partial L}{\partial \dot{X}^\mu} = -\frac{T_0}{c} \frac{(\dot{X} \cdot X')X'_\mu - (X')^2 \dot{X}_\mu}{\sqrt{\left(\dot{X} \cdot X'\right)^2 - \left(\dot{X}\right)^2 (X')^2}} \tag{19.10}$$

and

$$\mathcal{P}^\sigma_\mu = \frac{\partial L}{\partial X'^\mu} = -\frac{T_0}{c} \frac{(\dot{X} \cdot X')\dot{X}_\mu - (X')^2 X'_\mu}{\sqrt{\left(\dot{X} \cdot X'\right)^2 - \left(\dot{X}\right)^2 (X')^2}}. \tag{19.11}$$

In terms of them, the change in the action is

$$\delta S = \int_{\tau_i}^{\tau_f}\!\!\int_0^{\sigma_1} \left[\frac{\partial}{\partial \tau} \left(\delta X^\mu \mathcal{P}^\tau_\mu\right) + \frac{\partial}{\partial \sigma} \left(\delta X^\mu \mathcal{P}^\sigma_\mu\right) - \delta X^\mu \left(\frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma_\mu}{\partial \sigma} \right) \right] d\tau \, d\sigma. \tag{19.12}$$

The total τ-derivative integrates to a term involving the variation δX^μ which we require to vanish at the initial and final values of τ. So we drop that term and find that the net change in the action is

$$\delta S = \int_{\tau_i}^{\tau_f} \left[\delta X^\mu \mathcal{P}^\sigma_\mu\right]_0^{\sigma_1} d\tau - \int_{\tau_i}^{\tau_f}\!\!\int_0^{\sigma_1} \delta X^\mu \left(\frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma_\mu}{\partial \sigma} \right) d\tau \, d\sigma. \tag{19.13}$$

Thus the equations of motion for the string are

$$\frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma_\mu}{\partial \sigma} = 0, \tag{19.14}$$

but the action is stationary only if the boundary conditions

$$\delta X^\mu(\tau, \sigma_1)\mathcal{P}^\sigma_\mu(\tau, \sigma_1) = \delta X^\mu(\tau, 0)\mathcal{P}^\sigma_\mu(\tau, 0) = 0 \tag{19.15}$$

are satisfied for all τ and each μ. These are conditions on the ends of open strings; closed strings satisfy them automatically.

The boundary conditions (19.15) are $2D = 2(d+1)$ conditions — one for each end σ_* of the string and each dimension μ of space-time:

$$\delta X^\mu(\tau, \sigma_*)\mathcal{P}^\sigma_\mu(\tau, \sigma_*) = 0 \quad \text{no sum over } \mu. \tag{19.16}$$

A **Dirichlet boundary condition** fixes a spatial component at an end of the string by

$$\dot{X}^i(\tau, \sigma_*) = 0 \tag{19.17}$$

or equivalently by $\delta X^\mu(\tau, \sigma_*) = 0$. The time component X^0 can not have a vanishing τ derivative, so it must obey a **free-endpoint boundary condition**

$$\mathcal{P}^\sigma_\mu(\tau, \sigma_*) = 0 \tag{19.18}$$