using as the action the area

$$S = -\frac{T_0}{c} \int_{\tau_i}^{\tau_f} \int_0^{\sigma_1} \sqrt{\left(\dot{X} \cdot X'\right)^2 - \left(\dot{X}\right)^2 (X')^2}\, d\tau\, d\sigma \tag{19.2}$$

in which

$$\dot{X}^\mu = \frac{\partial X^\mu}{\partial \tau} \quad \text{and} \quad X'^\mu = \frac{\partial X^\mu}{\partial \sigma} \tag{19.3}$$

and a Lorentz metric $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, \dots)$ is used to form the inner products

$$\dot{X} \cdot X' = \dot{X}^\mu \eta_{\mu\nu} X^{\nu\prime} \quad etc. \tag{19.4}$$

This action is the area swept out by a string of length σ_1 in time $\tau_f - \tau_i$.

If $\dot{X} d\tau = dt$ points in the time direction and $X' d\sigma = d\boldsymbol{r}$ points in a spatial direction, then it is easy to see that $\dot{X} \cdot X' = 0$, that $-\left(\dot{X}\right)^2 d\tau^2 = dt^2$, and that $(X')^2\, d\sigma^2 = d\boldsymbol{r}^2$. So in this simple case, the action (19.2) is

$$S = -\frac{T_0}{c} \int_{t_i}^{t_f} \int_0^{r_1} dt\, dr = -\frac{T_0}{c}(t_f - t_i) r_1 \tag{19.5}$$

which is the area the string sweeps out. The other term within the square-root ensures that the action is the area swept out for all $\dot{X}$ and X', and that it is invariant under arbitrary reparametrizations $\sigma \to \sigma'$ and $\tau \to \tau'$.

The equation of motion for the relativistic string follows from the requirement that the action (19.2) be stationary, $\delta S = 0$. Since

$$\delta \dot{X}^\mu = \delta \frac{\partial X^\mu}{\partial \tau} = \frac{\partial (X^\mu + \delta X^\mu)}{\partial \tau} - \frac{\partial X^\mu}{\partial \tau} = \frac{\partial \delta X^\mu}{\partial \tau} \tag{19.6}$$

and similarly

$$\delta X'^\mu = \frac{\partial \delta X^\mu}{\partial \sigma} \tag{19.7}$$

we may express the change in the action in terms of derivatives of the Lagrange density

$$L = -\frac{T_0}{c} \sqrt{\left(\dot{X} \cdot X'\right)^2 - \left(\dot{X}\right)^2 (X')^2}. \tag{19.8}$$

as

$$\delta S = \int_{\tau_i}^{\tau_f} \int_0^{\sigma_1} \left[\frac{\partial L}{\partial \dot{X}^\mu} \frac{\partial \delta X^\mu}{\partial \tau} + \frac{\partial L}{\partial X'^\mu} \frac{\partial \delta X^\mu}{\partial \sigma} \right] d\tau\, d\sigma. \tag{19.9}$$