where for n_f flavors of light quarks

$$\beta_0 = \frac{1}{(4\pi)^2}\left(\frac{11}{3}N - \frac{2}{3}n_f\right)$$
$$\beta_1 = \frac{1}{(4\pi)^4}\left(\frac{34}{3}N^2 - \frac{10}{3}Nn_f - \frac{N^2-1}{N}n_f\right). \tag{17.33}$$

In quantum chromodynamics, $N = 3$.

Combining the definition (17.31) of the β-function with its expansion (17.32) for small g, one arrives at the differential equation

$$\frac{dg}{d\ln a} = \beta_0\, g^3 + \beta_1\, g^5 \tag{17.34}$$

which one may integrate

$$\int d\ln a = \ln a - \ln c = \int \frac{dg}{\beta_0 g^3 + \beta_1 g^5} = -\frac{1}{2\beta_0 g^2} + \frac{\beta_1}{2\beta_0^2}\ln\left(\frac{\beta_0 + \beta_1 g^2}{g^2}\right) \tag{17.35}$$

to find

$$a(g) = c\left(\frac{\beta_0 + \beta_1 g^2}{g^2}\right)^{\beta_1/2\beta_0^2} e^{-1/2\beta_0 g^2} \tag{17.36}$$

in which c is a constant of integration. The term $\beta_1 g^2$ is of higher order in g, and if one drops it and absorbs a power of β_0 into a new constant of integration Λ, then one gets

$$a(g) = \frac{1}{\Lambda}\left(\beta_0 g^2\right)^{-\beta_1/2\beta_0^2} e^{-1/2\beta_0 g^2}. \tag{17.37}$$

As $g \to 0$, the lattice spacing $a(g)$ goes to zero *very fast* (as long as $n_f < 17$ for $N = 3$). The inverse of this relation (17.37) is

$$g(a) \approx \left[\beta_0 \ln(a^{-2}\Lambda^{-2}) + (\beta_1/\beta_0)\ln\left(\ln(a^{-2}\Lambda^{-2})\right)\right]^{-1/2}. \tag{17.38}$$

It shows that the coupling constant slowly goes to zero with a, which is a lattice version of **asymptotic freedom**. □

17.3 The Renormalization Group in Condensed-Matter Physics

The study of condensed matter is concerned mainly with properties that emerge in the bulk, such as the melting point, the boiling point, or the conductivity. So we want to see what happens to the physics when we increase the distance scale many orders of magnitude beyond the size a of an individual molecule or the distance between nearest neighbors.