The full action of a stretched field is

$$S(\phi_L) = \int d^d x \left(\frac{1}{2}(\partial\phi)^2 + \sum_n g_{d,n}(L)\phi^n \right) \tag{17.48}$$

in which

$$g_{d,n}(L) = L^d A^n(L)\, g_n = L^{d-n(d-2)/2}\, g_{d,n}. \tag{17.49}$$

The beta-function

$$\beta(g_{d,n}) \equiv \frac{L}{g_{d,n}(L)} \frac{dg_{d,n}(L)}{dL} = d - n(d-2)/2 \tag{17.50}$$

is just the exponent of the coupling "constant" $g_{d,n}(L)$. If it is positive, then the coupling constant $g_{d,n}(L)$ gets stronger as $L \to \infty$; such couplings are called **relevant**. Couplings with vanishing exponents are insensitive to changes in L and are **marginal**. Those with negative exponents shrink with increasing L; they are **irrelevant**.

The coupling constant $g_{d,n,p}$ of a term with p derivatives and n powers of the field ϕ in a space of d dimensions varies as

$$g_{d,n,p}(L) = L^d\, A^n(L)\, L^{-p}\, g_{n,p} = L^{d-n(d-2)/2-p}\, g_{d,n,p}. \tag{17.51}$$

Example 17.3 (QCD) In quantum chromodynamics, there is a cubic term $g\, f_{abc}\, A_0^a A_i^b \partial_0 A_i^c$ which in effect looks like $g\, f_{abc}\, \phi_a \phi_b \dot{\phi}_c$. Is it relevant? Well, if we stretch space but not time, then the time derivative has no effect, and $d = 3$. So the cubic, $n = 3$, grows as $L^{3/2}$

$$g_{3,3,0}(L) = L^{d-n(d-2)/2}\, g_{3,3,0} = L^{3/2}\, g_{3,3,0}. \tag{17.52}$$

Since this cubic term drives asymptotic freedom, its strengthening as space is stretched by the dimensionless factor L may point to a qualitative explanation of confinement. For if $g_{3,3,0}(L)$ grows with distance as $L^{3/2}$, then $\alpha_s(L) = g_{3,3,0}^2(L)/4\pi$ grows as L^3, and so the strength $\alpha_s(Lr)/(Lr)^2$ of the force between two quarks separated by a distance Lr grows linearly with L

$$F(Lr) = \frac{\alpha_s(Lr)}{(Lr)^2} = \frac{L^3\, \alpha_s(r)}{(Lr)^2} = L\, \frac{\alpha_s(r)}{r^2} \tag{17.53}$$

which may be enough for quark confinement.

On the other hand, if we stretch both space and time, then the cubic $g_{4,3,1}(L)$ and quartic $g_{4,4,0}(L)$ couplings are marginal. □