17

The Renormalization Group

17.1 The Renormalization Group in Quantum Field Theory

Most quantum field theories are non-linear with infinitely many degrees of freedom, and because they describe point particles, they are rife with infinities. But short-distance effects, probably the finite sizes of the fundamental constituents of matter, mitigate these infinities so that we can cope with them consistently without knowing what happens at very short distances and very high energies. This procedure is called **renormalization**.

For instance, in the theory described by the Lagrange density

$$\mathcal{L} = -\frac{1}{2}\partial_\nu \phi\, \partial^\nu \phi - \frac{1}{2}m^2\phi^2 - \frac{g}{24}\phi^4 \tag{17.1}$$

we can cut off divergent integrals at some high energy Λ. The amplitude for the elastic scattering of two bosons of initial four-momenta p_1 and p_2 into two of final momenta p_1' and p_2' to one-loop order (Weinberg, 1996, chap. 18) then is proportional to (Zee, 2010, chaps. III & VI)

$$A = g - \frac{g^2}{32\pi^2}\left[\ln\left(\frac{\Lambda^6}{stu}\right) - i\pi + 3\right] \tag{17.2}$$

as long as the absolute values of the Mandelstam variables $s = -(p_1 + p_2)^2$, $t = -(p_1 - p_1')^2$, and $u = -(p_1 - p_2')^2$, which satisfy $stu > 0$ and $s + t + u = 4m^2$, are all much larger than m^2 (Stanley Mandelstam, 1928–). We define the **physical coupling constant** g_μ, as opposed to the **bare** one g that comes with $\mathcal{L}$, to be the real part of the amplitude A at $s = -t = -u = \mu^2$

$$g_\mu = g - \frac{3g^2}{32\pi^2}\left[\ln\left(\frac{\Lambda^2}{\mu^2}\right) + 1\right]. \tag{17.3}$$

Thus the bare coupling constant is $g = g_\mu + 3g^2\left[\ln(\Lambda^2/\mu^2) + 1\right]$, and using this formula, we can write our expression (17.2) for the amplitude A in a form in which the cutoff Λ no longer appears

$$A = g_\mu - \frac{g^2}{32\pi^2}\left[\ln\left(\frac{\mu^6}{stu}\right) - i\pi\right]. \tag{17.4}$$

This is the magic of renormalization.

The physical coupling "constant" g_μ is the right coupling at energy μ because when all the Mandelstam variables are near the renormalization point $stu = \mu^6$, the one-loop correction is tiny, and $A \approx g_\mu$.

How does the physical coupling g_μ depend upon the energy μ? The amplitude A must be independent of the renormalization energy μ, and so

$$\frac{dA}{d\mu} = \frac{dg_\mu}{d\mu} - \frac{g^2}{32\pi^2}\frac{6}{\mu} = 0 \tag{17.5}$$

which is a version of the **Callan-Symanzik equation**.

We assume that when the cutoff Λ is big but finite, the bare and **running** coupling constants g and g_μ are so tiny that they differ by terms of order g^2 or g_μ^2. Then to lowest order in g and g_μ, we can replace g^2 by g_μ^2 in (17.5) and arrive at the simple differential equation

$$\mu\,\frac{dg_\mu}{d\mu} \equiv \beta(g_\mu) = \frac{3\,g_\mu^2}{16\pi^2} \tag{17.6}$$

which we can integrate

$$\ln\frac{E}{M} = \int_M^E \frac{d\mu}{\mu} = \int_{g_M}^{g_E} \frac{dg_\mu}{\beta(g_\mu)} = \frac{16\pi^2}{3}\int_{g_M}^{g_E}\frac{dg_\mu}{g_\mu^2} = \frac{16\pi^2}{3}\left(\frac{1}{g_M} - \frac{1}{g_E}\right) \tag{17.7}$$

to find the running physical coupling constant g_μ at energy $\mu = E$

$$g_E = \frac{g_M}{1 - 3\,g_M\,\ln(E/M)/16\pi^2}. \tag{17.8}$$

As the energy $E = \sqrt{s}$ rises above M, while staying below the singular value $E = M\exp(16\pi^2/3g_M)$, the running coupling g_E slowly increases. And so does the scattering amplitude, $A \approx g_E$.

Example 17.1 (Quantum Electrodynamics) Vacuum polarization makes the amplitude for the scattering of two electrons proportional to (Weinberg, 1995, chap. 11)

$$A(q^2) = e^2\,\left[1 + \pi(q^2)\right] \tag{17.9}$$

rather than to e^2. Here e is the renormalized charge, $q = p_1' - p_1$ is the four-momentum transferred to the first electron, and

$$\pi(q^2) = \frac{e^2}{2\pi^2} \int_0^1 x(1-x) \ln\left[1 + \frac{q^2 x(1-x)}{m^2}\right] dx \tag{17.10}$$

represents the polarization of the vacuum. We define the square of the running coupling constant e_μ^2 to be the amplitude (17.9) at $q^2 = \mu^2$

$$e_\mu^2 = A(\mu^2) = e^2 \left[1 + \pi(\mu^2)\right]. \tag{17.11}$$

For $\mu^2 \gg m^2$, the vacuum polarization term $\pi(\mu^2)$ is (exercise 17.1)

$$\pi(\mu^2) \approx \frac{e^2}{6\pi^2}\left[\ln\frac{\mu}{m} - \frac{5}{6}\right]. \tag{17.12}$$

The amplitude (17.9) then is

$$A(q^2) = e_\mu^2 \, \frac{1+\pi(q^2)}{1+\pi(\mu^2)} \tag{17.13}$$

and since it must be independent of μ, we have

$$0 = \frac{d}{d\mu}\frac{A(q^2)}{1+\pi(q^2)} = \frac{d}{d\mu}\,\frac{e_\mu^2}{1+\pi(\mu^2)} \approx \frac{d}{d\mu}\left\{e_\mu^2 \left[1 - \pi(\mu^2)\right]\right\}. \tag{17.14}$$

So we find

$$0 = 2e_\mu \left(\frac{de_\mu}{d\mu}\right)\left[1-\pi(\mu^2)\right] - e_\mu^2 \frac{d\pi(\mu^2)}{d\mu} = 2e_\mu \left(\frac{de_\mu}{d\mu}\right)\left[1-\pi(\mu^2)\right] - e_\mu^2 \frac{e^2}{6\pi^2\mu}. \tag{17.15}$$

Thus since by (17.10 & 17.11) $\pi(\mu^2) = \mathcal{O}(e^2)$ and $e_\mu^2 = e^2 + \mathcal{O}(e^4)$, we find to lowest order in e_μ

$$\mu\frac{de_\mu}{d\mu} \equiv \beta(e_\mu) = \frac{e_\mu^3}{12\pi^2}. \tag{17.16}$$

We can integrate this differential equation

$$\ln\frac{E}{M} = \int_M^E \frac{d\mu}{\mu} = \int_{e_M}^{e_E} \frac{de_\mu}{\beta(e_\mu)} = 12\pi^2 \int_{e_M}^{e_E} \frac{de_\mu}{e_\mu^3} = 6\pi^2\left(\frac{1}{e_M^2} - \frac{1}{e_E^2}\right) \tag{17.17}$$

and so get for the running coupling constant the formula

$$e_E^2 = \frac{e_M^2}{1 - e_M^2 \, \ln(E/M)/6\pi^2} \tag{17.18}$$

which shows that it slowly increases with the energy E. Thus, the fine-structure constant $e_\mu^2/4\pi$ rises from $\alpha = 1/137.036$ at m_e to

$$\frac{e^2(45.5\text{GeV})}{4\pi} = \frac{\alpha}{1 - 2\alpha\ln(45.5/0.00051)/3\pi} = \frac{1}{134.6} \tag{17.19}$$