then (exercise 16.26) the state

\[ |\chi\rangle = \exp \left( \int \sum_m \psi_m^\dagger(x, 0) \chi_m(x) - \frac{1}{2} \chi_m^\dagger(x) \chi_m(x) \, d^3x \right) |0\rangle \]

\[ = \exp \left( \int \psi^\dagger \chi - \frac{1}{2} \chi^\dagger \chi \, d^3x \right) |0\rangle \]  

(16.211)
is an eigenstate of the operator \( \psi_m(x, 0) \) with eigenvalue \( \chi_m(x) \)

\[ \psi_m(x, 0) |\chi\rangle = \chi_m(x) |\chi\rangle. \]  

(16.212)
The inner product of two such states is (exercise 16.27)

\[ \langle \chi' | \chi \rangle = \exp \left[ \int \chi'^\dagger \chi - \frac{1}{2} \chi'^\dagger \chi' - \frac{1}{2} \chi^\dagger \chi \, d^3x \right]. \]  

(16.213)
The identity operator is the integral

\[ I = \int |\chi\rangle \langle \chi| \, D\chi^* D\chi \]  

(16.214)in which

\[ D\chi^* D\chi = \prod_{m,x} d\chi_m^*(x) d\chi_m(x). \]  

(16.215)
The hamiltonian for a free Dirac field \( \psi \) of mass \( m \) is the spatial integral

\[ H_0 = \int \bar{\psi} (\gamma \cdot \nabla + m) \psi \, d^3x \]  

(16.216)in which \( \bar{\psi} \equiv i \psi^\dagger \gamma^0 \) and the gamma matrices (10.287) satisfy

\[ \{ \gamma^a, \gamma^b \} = 2 \eta^{ab} \]  

(16.217)where \( \eta \) is the \( 4 \times 4 \) diagonal matrix with entries \((-1, 1, 1, 1)\). Since \( \psi |\chi\rangle = \chi |\chi\rangle \) and \( \langle \chi' | \psi^\dagger = \langle \chi' | \chi'^\dagger \), the quantity \( \langle \chi' | \exp(-i\epsilon H_0) |\chi\rangle \) is by (16.213)

\[ \langle \chi' | e^{-i\epsilon H_0} |\chi\rangle = \langle \chi' | \chi \rangle \exp \left[ -i \epsilon \int \bar{\chi}' (\gamma \cdot \nabla + m) \chi \, d^3x \right] \]

\[ = \exp \left[ \int \frac{1}{2} (\chi'^\dagger - \chi^\dagger) \chi - \frac{1}{2} \chi'^\dagger (\chi' - \chi) - i\epsilon \bar{\chi}' (\gamma \cdot \nabla + m) \chi \, d^3x \right] \]

\[ = \exp \left\{ \epsilon \int \left[ \frac{1}{2} \chi'^\dagger - \frac{1}{2} \chi^\dagger \chi' - i\bar{\chi}' (\gamma \cdot \nabla + m) \chi \right] \, d^3x \right\} \]

in which \( \chi'^\dagger - \chi^\dagger = \epsilon \chi'^\dagger \) and \( \chi' - \chi = \epsilon \chi \). Everything within the square
brackets is multiplied by $\epsilon$, so we may replace $\chi^\dagger$ by $\chi^\dagger$ and $\bar{\chi}$ by $\bar{\chi}$ so as to write to first order in $\epsilon$
\[
\langle \chi' | e^{-i\epsilon H_0} | \chi \rangle = \exp \left[ \epsilon \int \frac{1}{2} \hat{\chi}^\dagger \hat{\chi} - \frac{1}{2} \hat{\chi}^\dagger \hat{\psi} - i \hat{\chi} (\gamma \cdot \nabla + m) \chi \right] d^3x \tag{16.219}
\]
in which the dependence upon $\chi'$ is through the time derivatives.

Putting together $n = 2t/\epsilon$ such matrix elements, integrating over all intermediate-state dyadics $|\chi\rangle \langle \chi|$, and using our formula (16.214), we find
\[
\langle \chi_t | e^{-2itH_0} | \chi_{-t} \rangle = \int \exp \left[ \int \frac{1}{2} \hat{\chi}^\dagger \chi - \frac{1}{2} \hat{\chi}^\dagger \hat{\psi} - i \hat{\chi} (\gamma \cdot \nabla + m) \chi d^4x \right] D\chi^* D\chi. \tag{16.220}
\]

Integrating $\hat{\chi}^\dagger \hat{\chi}$ by parts and dropping the surface term, we get
\[
\langle \chi_t | e^{-2itH_0} | \chi_{-t} \rangle = \int \exp \left[ \int - \hat{\chi}^\dagger \hat{\psi} - i \hat{\chi} (\gamma \cdot \nabla + m) \chi d^4x \right] D\chi^* D\chi. \tag{16.221}
\]

Since $- \hat{\chi}^\dagger \hat{\psi} = - i \bar{\chi}^0 \hat{\chi}$, the argument of the exponential is
\[
i \int - \bar{\chi}^0 \hat{\chi} - \bar{\chi} (\gamma \cdot \nabla + m) \chi d^4x = i \int - \bar{\chi} (\gamma^\mu \partial_\mu + m) \chi d^4x. \tag{16.222}
\]

We then have
\[
\langle \chi_t | e^{-2itH_0} | \chi_{-t} \rangle = \int \exp \left( i \int \mathcal{L}_0(\chi) d^4x \right) D\chi^* D\chi \tag{16.223}
\]
in which $\mathcal{L}_0(\chi) = - \bar{\chi} (\gamma^\mu \partial_\mu + m) \chi$ is the action density (10.289) for a free Dirac field. Thus the amplitude is a path integral with phases given by the classical action $S_0[\chi]$
\[
\langle \chi_t | e^{-2itH_0} | \chi_{-t} \rangle = \int e^{i \int \mathcal{L}_0(\chi) d^4x} D\chi^* D\chi = \int e^{i S_0[\chi]} D\chi^* D\chi \tag{16.224}
\]
and the integral is over all fields that go from $\chi(x, -t) = \chi_{-t}(x)$ to $\chi(x, t) = \chi_t(x)$. Any normalization factor will cancel in ratios of such integrals.

Since Fermi fields anticommute, their time-ordered product has an extra minus sign
\[
\mathcal{T} \left[ \bar{\psi}(x_1) \psi(x_2) \right] = \theta(x_1^0 - x_2^0) \bar{\psi}(x_1) \psi(x_2) - \theta(x_2^0 - x_1^0) \psi(x_2) \bar{\psi}(x_1). \tag{16.225}
\]

The logic behind our formulas (16.122) and (16.128) for the time-ordered product of bosonic fields now leads to an expression for the time-ordered