then (exercise 16.26) the state

$$|\chi\rangle = \exp\left(\int \sum_m \psi_m^\dagger(\boldsymbol{x},0)\,\chi_m(\boldsymbol{x}) - \frac{1}{2}\chi_m^*(\boldsymbol{x})\chi_m(\boldsymbol{x})\,d^3x\right)|0\rangle$$
$$= \exp\left(\int \psi^\dagger\chi - \tfrac{1}{2}\chi^\dagger\chi\,d^3x\right)|0\rangle \tag{16.211}$$

is an eigenstate of the operator $\psi_m(\boldsymbol{x},0)$ with eigenvalue $\chi_m(\boldsymbol{x})$

$$\psi_m(\boldsymbol{x},0)|\chi\rangle = \chi_m(\boldsymbol{x})|\chi\rangle. \tag{16.212}$$

The inner product of two such states is (exercise 16.27)

$$\langle\chi'|\chi\rangle = \exp\left[\int \chi'^\dagger\chi - \frac{1}{2}\chi'^\dagger\chi' - \frac{1}{2}\chi^\dagger\chi\; d^3x\right]. \tag{16.213}$$

The identity operator is the integral

$$I = \int |\chi\rangle\langle\chi|\, D\chi^* D\chi \tag{16.214}$$

in which

$$D\chi^* D\chi \equiv \prod_{m,\boldsymbol{x}} d\chi_m^*(\boldsymbol{x}) d\chi_m(\boldsymbol{x}). \tag{16.215}$$

The hamiltonian for a free Dirac field ψ of mass m is the spatial integral

$$H_0 = \int \overline{\psi}\,(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m)\,\psi\; d^3x \tag{16.216}$$

in which $\overline{\psi} \equiv i\psi^\dagger\gamma^0$ and the gamma matrices (10.287) satisfy

$$\{\gamma^a, \gamma^b\} = 2\,\eta^{ab} \tag{16.217}$$

where η is the 4×4 diagonal matrix with entries $(-1,1,1,1)$. Since $\psi|\chi\rangle = \chi|\chi\rangle$ and $\langle\chi'|\psi^\dagger = \langle\chi'|\chi'^\dagger$, the quantity $\langle\chi'|\exp(-i\epsilon H_0)|\chi\rangle$ is by (16.213)

$$\langle\chi'|e^{-i\epsilon H_0}|\chi\rangle = \langle\chi'|\chi\rangle \exp\left[-i\epsilon\int \overline{\chi}'\,(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m)\,\chi\; d^3x\right] \tag{16.218}$$
$$= \exp\left[\int \tfrac{1}{2}(\chi'^\dagger - \chi^\dagger)\chi - \tfrac{1}{2}\chi'^\dagger(\chi' - \chi) - i\epsilon\overline{\chi}'(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m)\chi d^3x\right]$$
$$= \exp\left\{\epsilon\int\left[\tfrac{1}{2}\dot{\chi}^\dagger\chi - \tfrac{1}{2}\chi'^\dagger\dot{\chi} - i\overline{\chi}'\,(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m)\,\chi\right]\; d^3x\right\}$$

in which $\chi'^\dagger - \chi^\dagger = \epsilon\dot{\chi}^\dagger$ and $\chi' - \chi = \epsilon\dot{\chi}$. Everything within the square brackets is multiplied by ϵ, so we may replace $\chi'^\dagger$ by $\chi^\dagger$ and $\overline{\chi}'$ by $\overline{\chi}$ so as to write to first order in ϵ

$$\langle\chi'|e^{-i\epsilon H_0}|\chi\rangle = \exp\left[\epsilon\int \tfrac{1}{2}\dot{\chi}^\dagger\chi - \tfrac{1}{2}\chi^\dagger\dot{\chi} - i\overline{\chi}\,(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m)\,\chi\; d^3x\right] \tag{16.219}$$

in which the dependence upon χ' is through the time derivatives.

Putting together $n = 2t/\epsilon$ such matrix elements, integrating over all intermediate-state dyadics $|\chi\rangle\langle\chi|$, and using our formula (16.214), we find

$$\langle\chi_t|e^{-2itH_0}|\chi_{-t}\rangle = \int \exp\left[\int \tfrac{1}{2}\dot{\chi}^\dagger\chi - \tfrac{1}{2}\chi^\dagger\dot{\chi} - i\overline{\chi}\,(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m)\,\chi d^4x\right] D\chi^* D\chi. \tag{16.220}$$

Integrating $\dot{\chi}^\dagger\chi$ by parts and dropping the surface term, we get

$$\langle\chi_t|e^{-2itH_0}|\chi_{-t}\rangle = \int \exp\left[\int -\,\chi^\dagger\dot{\chi} - i\overline{\chi}\,(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m)\,\chi\, d^4x\right] D\chi^* D\chi. \tag{16.221}$$

Since $-\,\chi^\dagger\dot{\chi} = -\,i\overline{\chi}\gamma^0\dot{\chi}$, the argument of the exponential is

$$i\int -\,\overline{\chi}\gamma^0\dot{\chi} - \overline{\chi}\,(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m)\,\chi\; d^4x = i\int -\,\overline{\chi}\,(\gamma^\mu\partial_\mu + m)\,\chi\; d^4x. \tag{16.222}$$

We then have

$$\langle\chi_t|e^{-2itH_0}|\chi_{-t}\rangle = \int \exp\left(i\int \mathcal{L}_0(\chi)\, d^4x\right) D\chi^* D\chi \tag{16.223}$$

in which $\mathcal{L}_0(\chi) = -\,\overline{\chi}\,(\gamma^\mu\partial_\mu + m)\,\chi$ is the action density (10.289) for a free Dirac field. Thus the amplitude is a path integral with phases given by the classical action $S_0[\chi]$

$$\langle\chi_t|e^{-2itH_0}|\chi_{-t}\rangle = \int e^{i\int\mathcal{L}_0(\chi)\,d^4x} D\chi^* D\chi = \int e^{iS_0[\chi]} D\chi^* D\chi \tag{16.224}$$

and the integral is over all fields that go from $\chi(\boldsymbol{x}, -t) = \chi_{-t}(\boldsymbol{x})$ to $\chi(\boldsymbol{x}, t) = \chi_t(\boldsymbol{x})$. Any normalization factor will cancel in ratios of such integrals.

Since Fermi fields anticommute, their time-ordered product has an extra minus sign

$$\mathcal{T}\left[\overline{\psi}(x_1)\psi(x_2)\right] = \theta(x_1^0 - x_2^0)\,\overline{\psi}(x_1)\,\psi(x_2) - \theta(x_2^0 - x_1^0)\,\psi(x_2)\,\overline{\psi}(x_1). \tag{16.225}$$

The logic behind our formulas (16.122) and (16.128) for the time-ordered product of bosonic fields now leads to an expression for the time-ordered