and the functional delta function
\[
\delta[\nabla \cdot A] = \prod_x \delta(\nabla \cdot A(x)) \tag{16.163}
\]
enforces the Coulomb-gauge condition. The term \(\mathcal{L}_m\) is the action density of the matter field \(\psi\).

Tricks are available. We introduce a new field \(A^0(x)\) and consider the factor
\[
F = \int \exp \left[ i \int \frac{1}{2} \left( \nabla A^0 + \nabla \Delta^{-1} j^0 \right)^2 d^4x \right] DA^0 \tag{16.164}
\]
which is just a number independent of the charge density \(j^0\) since we can cancel the \(j^0\) term by shifting \(A^0\). By \(\Delta^{-1}\), we mean \(-1/4\pi |x - y|\). By integrating by parts, we can write the number \(F\) as (exercise 16.21)
\[
F = \int \exp \left[ i \int \frac{1}{2} \left( \nabla A^0 \right)^2 - A^0 j^0 - \frac{1}{2} j^0 \Delta^{-1} j^0 d^4x \right] DA^0
\]
\[
= \int \exp \left[ i \int \frac{1}{2} \left( \nabla A^0 \right)^2 - A^0 j^0 d^4x + i \int V_C dt \right] DA^0. \tag{16.165}
\]

So when we multiply the numerator and denominator of the amplitude (16.161) by \(F\), the awkward Coulomb term cancels, and we get
\[
\langle \Omega | T[O_1 \ldots O_n] | \Omega \rangle = \frac{\int O_1 \ldots O_n e^{iS'} \delta[\nabla \cdot A] DA D\psi}{\int e^{iS'} \delta[\nabla \cdot A] DA D\psi} \tag{16.166}
\]
where now \(DA\) includes all four components \(A^\mu\) and
\[
S' = \int \frac{1}{2} \dot{A}^2 - \frac{1}{2} (\nabla \times A)^2 + \frac{1}{2} (\nabla A^0)^2 + A \cdot j - A^0 j^0 + \mathcal{L}_m d^4x. \tag{16.167}
\]

Since the delta-function \(\delta[\nabla \cdot A]\) enforces the Coulomb-gauge condition, we can add to the action \(S'\) the term \((\nabla \cdot \dot{A}) A^0\) which is \(-\dot{A} \cdot \nabla A^0\) after we integrate by parts and drop the surface term. This extra term makes the action gauge invariant
\[
S = \int \frac{1}{2} (\dot{A} - \nabla A^0)^2 - \frac{1}{2} (\nabla \times A)^2 + A \cdot j - A^0 j^0 + \mathcal{L}_m d^4x
\]
\[
= \int -\frac{1}{2} F_{ab} F^{ab} + A^b j_b + \mathcal{L}_m d^4x. \tag{16.168}
\]