Linking three of these matrix elements together and using subscripts instead of primes, we have

$$\langle q_3|e^{-3\epsilon H}|q_0\rangle = \iint_{-\infty}^{\infty} \langle q_3|e^{-\epsilon H}|q_2\rangle\langle q_2|e^{-\epsilon H}|q_1\rangle\langle q_1|e^{-\epsilon H}|q_0\rangle \, dq_1 dq_2 \qquad (16.23)$$

$$= \left(\frac{m}{2\pi\epsilon}\right)^{3/2} \iint_{-\infty}^{\infty} \exp\left\{ -\epsilon \sum_{j=0}^{2} \left[\tfrac{1}{2}\, m\, \dot{q}_j^2 + V(q_j) \right] \right\} dq_1 dq_2.$$

Boldly passing from 3 to n and suppressing some integral signs, we get

$$\langle q_n|e^{-n\epsilon H}|q_0\rangle = \iiint_{-\infty}^{\infty} \langle q_n|e^{-\epsilon H}|q_{n-1}\rangle \ldots \langle q_1|e^{-\epsilon H}|q_0\rangle \, dq_1 \ldots dq_{n-1} \qquad (16.24)$$

$$= \left(\frac{m}{2\pi\epsilon}\right)^{n/2} \iiint_{-\infty}^{\infty} \exp\left\{ -\epsilon \sum_{j=0}^{n-1} \left[\tfrac{1}{2}\, m\, \dot{q}_j^2 + V(q_j) \right] \right\} dq_1 \ldots dq_{n-1}.$$

Writing dt for ϵ and taking the limits $\epsilon \to 0$ and $n \equiv \beta/\epsilon \to \infty$, we find that the matrix element $\langle q_\beta|e^{-\beta H}|q_0\rangle$ is a path integral of the exponential of the average energy multiplied by $-\beta$

$$\langle q_\beta|e^{-\beta H}|q_0\rangle = \int \exp\left[-\int_0^\beta \tfrac{1}{2} m \dot{q}^2(t) + V(q(t)) \, dt \right] Dq \qquad (16.25)$$

in which $Dq \equiv (n\, m/2\pi\, \beta)^{n/2} \, dq_1 dq_2 \cdots dq_{n-1}$ as $n \to \infty$. We sum over all paths $q(t)$ that go from $q(0) = q_0$ at inverse temperature $\beta = 0$ to $q(\beta) = q_\beta$ at inverse temperature β.

In the limit $\beta \to \infty$, the operator $\exp(-\beta H)$ becomes proportional to a projection operator (16.1) on the ground state of the theory.

In three-dimensions with $\dot{\boldsymbol{q}}(\beta) = d\boldsymbol{q}(\beta)/d\beta$, and $\hbar \neq 1$, equation (16.25) becomes (exercise 16.28)

$$\langle \boldsymbol{q}_\beta|e^{-\beta H}|\boldsymbol{q}_0\rangle = \int \exp\left[-\int_0^\beta \frac{m}{2\hbar^2}\, \dot{\boldsymbol{q}}^2(\beta') + V(\boldsymbol{q}(\beta')) \, d\beta' \right] D\boldsymbol{q} \qquad (16.26)$$

where $Dq \equiv (n\, m/2\pi\, \beta\, \hbar^2)^{3n/2} \, dq_1 dq_2 \cdots dq_{n-1}$ as $n \to \infty$.

Path integrals in imaginary time are called *euclidean* mainly to distinguish them from *Minkowski* path integrals, which represent matrix elements of the time-evolution operator $\exp(-itH)$ in real time.