The spatial Fourier transform $\tilde{\phi}'(\boldsymbol{p})$

$$\phi'(\boldsymbol{x}) = \int e^{i\boldsymbol{p}\cdot\boldsymbol{x}}\, \tilde{\phi}'(\boldsymbol{p})\, \frac{d^3p}{(2\pi)^3} \tag{15.52}$$

satisfies $\tilde{\phi}'(-\boldsymbol{p}) = \tilde{\phi}'^*(\boldsymbol{p})$ since ϕ' is real. In terms of it, the ground-state wave function is

$$\langle\phi'|0\rangle = N \exp\left(-\frac{1}{2}\int |\tilde{\phi}'(\boldsymbol{p})|^2 \sqrt{\boldsymbol{p}^2 + m^2}\, \frac{d^3p}{(2\pi)^3}\right). \tag{15.53}$$

Example 15.3 (Other Theories, Other Vacua) We can find exact ground states for interacting theories with hamiltonians like

$$H = \frac{1}{2}\int \left[\sqrt{-\nabla^2 + m^2}\,\phi + c_n\phi^n - i\pi\right]\left[\sqrt{-\nabla^2 + m^2}\,\phi + c_n\phi^n + i\pi\right] d^3x. \tag{15.54}$$

The state $|\Omega\rangle$ will be an eigenstate of H with eigenvalue zero if

$$\frac{\delta\langle\phi'|\Omega\rangle}{\delta\phi'(\boldsymbol{x})} = -\left[\sqrt{-\nabla^2 + m^2}\,\phi'(\boldsymbol{x}) + c_n\phi'^n\right]\langle\phi'|\Omega\rangle. \tag{15.55}$$

By extending the argument of equations (15.45–15.51), one may show (exercise 15.5) that the wave functional of the vacuum is

$$\langle\phi'|\Omega\rangle = N \exp\left[-\int\left(\tfrac{1}{2}\phi'\sqrt{-\nabla^2 + m^2}\,\phi' + \frac{c_n}{n+1}\phi'^{n+1}\right) d^3x\right] \tag{15.56}$$

which is normalizable only when n is odd. □

Exercises

15.1 Compute the action $S_0[q]$ (15.1) for the classical path (15.24).

15.2 Use (15.25) to find a formula for the second functional derivative of the action (15.2) of the harmonic oscillator for which $V(q) = m\omega^2 q^2/2$.

15.3 Derive (15.53) from equations (15.48 & 15.52).

15.4 Show that the functional $\langle A'|0\rangle$ (**??**) satisfies (**??**).

15.5 Show that (15.56) satisfies (15.55).