The central limit theorem tells us that the distribution
\[ P^{(N)}(y) = \int 3x_1^2 3x_2^2 \ldots 3x_N^2 \delta((x_1 + x_2 + \cdots + x_N)/N - y) \, d^N x \] (13.234)
of the mean \( y = (x_1 + \cdots + x_N)/N \) tends as \( N \to \infty \) to Gauss’s distribution
\[
\lim_{N \to \infty} P^{(N)}(y) = \frac{1}{\sigma_y \sqrt{2\pi}} \exp\left(-\frac{(x - \mu_y)^2}{2\sigma_y^2}\right) \tag{13.235}
\]
with mean \( \mu_y \) and variance \( \sigma_y^2 \) given by (13.219). Since the \( P_j \)'s are all the same, they all have the same mean
\[
\mu_y = \mu_j = \int_0^1 3x^3 \, dx = \frac{3}{4} \tag{13.236}
\]
and the same variance
\[
\sigma_y^2 = \int_0^1 3x^4 \, dx - \left(\frac{3}{4}\right)^2 = \frac{3}{5} - \frac{9}{16} = \frac{3}{80} \tag{13.237}
\]
By (13.219), the variance of the mean \( y \) is then \( \sigma_y^2 = 3/80N \). Thus as \( N \) increases, the mean \( y \) tends to a gaussian with mean \( \mu_y = 3/4 \) and ever narrower peaks.

For \( N = 1 \), the probability distribution \( P^{(1)}(y) \) is
\[
P^{(1)}(y) = \int 3x_1^2 \delta(x_1 - y) \, dx_1 = 3y^2 \tag{13.238}
\]
which is the probability distribution we started with. In Fig. 13.5, this is the quadratic, dotted curve.

For \( N = 2 \), the probability distribution \( P^{(1)}(y) \) is (exercise 13.31)
\[
P^{(2)}(y) = \int 3x_1^2 3x_2^2 \delta((x_1 + x_2)/2 - y) \, dx_1 \, dx_2 \tag{13.239}
\]
\[= \theta\left(\frac{1}{2} - y\right) \frac{96}{5} y^5 + \theta(y - \frac{1}{2}) \left(\frac{36}{5} - \frac{96}{5} y^5 + 48y^2 - 36y\right).\]

You can get the probability distributions \( P^{(N)}(y) \) for \( N = 2^j \) by running the FORTAN95 program

```fortran
program clt
  implicit none ! avoids typos
  character(len=1)::ch_i1
  integer,parameter::dp = kind(1.d0) !define double precision
  integer::j,k,n,m
  integer,dimension(100)::plot = 0
end program clt
```