The central limit theorem tells us that the distribution

$$P^{(N)}(y) = \int 3x_1^2 \, 3x_2^2 \, \dots \, 3x_N^2 \, \delta((x_1 + x_2 + \dots + x_N)/N - y) \, d^N x \quad (13.234)$$

of the mean $y = (x_1 + \dots + x_N)/N$ tends as $N \to \infty$ to Gauss's distribution

$$\lim_{N\to\infty} P^{(N)}(y) = \frac{1}{\sigma_y \sqrt{2\pi}} \exp\left(- \frac{(x - \mu_y)^2}{2\sigma_y^2} \right) \quad (13.235)$$

with mean μ_y and variance σ_y^2 given by (13.219). Since the P_j's are all the same, they all have the same mean

$$\mu_y = \mu_j = \int_0^1 3x^3 dx = \frac{3}{4} \quad (13.236)$$

and the same variance

$$\sigma_j^2 = \int_0^1 3x^4 dx - \left(\frac{3}{4}\right)^2 = \frac{3}{5} - \frac{9}{16} = \frac{3}{80}. \quad (13.237)$$

By(13.219), the variance of the mean y is then $\sigma_y^2 = 3/80N$. Thus as N increases, the mean y tends to a gaussian with mean $\mu_y = 3/4$ and ever narrower peaks.

For $N = 1$, the probability distribution $P^{(1)}(y)$ is

$$P^{(1)}(y) = \int 3x_1^2 \, \delta(x_1 - y) \, dx_1 = 3y^2 \quad (13.238)$$

which is the probability distribution we started with. In Fig. 13.5, this is the quadratic, dotted curve.

For $N = 2$, the probability distribution $P^{(1)}(y)$ is (exercise 13.31)

$$\begin{aligned} P^{(2)}(y) &= \int 3x_1^2 \, 3x_2^2 \, \delta((x_1 + x_2)/2 - y) \, dx_1 \, dx_2 \quad (13.239) \\ &= \theta(\tfrac{1}{2} - y) \, \frac{96}{5} \, y^5 + \theta(y - \tfrac{1}{2}) \left(\frac{36}{5} - \frac{96}{5} \, y^5 + 48y^2 - 36y \right). \end{aligned}$$

You can get the probability distributions $P^{(N)}(y)$ for $N = 2^j$ by running the FORTAN95 program

```
program clt
  implicit none ! avoids typos
  character(len=1)::ch_i1
  integer,parameter::dp = kind(1.d0) !define double precision
  integer::j,k,n,m
  integer,dimension(100)::plot = 0
```