If we substitute our formula (13.169) for $\langle \boldsymbol{v}^2(t) \rangle$ into the expression (13.123) for the acceleration of $\langle \boldsymbol{r}^2 \rangle$, then we get

$$\frac{d^2 \langle \boldsymbol{r}^2(t) \rangle}{dt^2} = -\frac{1}{\tau} \frac{d\langle \boldsymbol{r}^2(t) \rangle}{dt} + 2e^{-2t/\tau} \langle \boldsymbol{v}^2(0) \rangle + \frac{6kT}{m} \left(1 - e^{-2t/\tau}\right). \quad (13.171)$$

The solution with both $\langle \boldsymbol{r}^2(0) \rangle = 0$ and $d\langle \boldsymbol{r}^2(0) \rangle / dt = 0$ is (exercise 13.21)

$$\langle \boldsymbol{r}^2(t) \rangle = \langle \boldsymbol{v}^2(0) \rangle \, \tau^2 \left(1 - e^{-t/\tau}\right)^2 - \frac{3kT}{m} \, \tau^2 \left(1 - e^{-t/\tau}\right)\left(3 - e^{-t/\tau}\right) + \frac{6kT\tau}{m} \, t. \quad (13.172)$$

13.12 Characteristic and Moment-Generating Functions

The Fourier transform (3.9) of a probability distribution $P(x)$ is its **characteristic function** $\tilde{P}(k)$ sometimes written as $\chi(k)$

$$\tilde{P}(k) \equiv \chi(k) \equiv E[e^{ikx}] = \int e^{ikx} \, P(x) \, dx. \quad (13.173)$$

The probability distribution $P(x)$ is the inverse Fourier transform (3.9)

$$P(x) = \int e^{-ikx} \, \tilde{P}(k) \, \frac{dk}{2\pi}. \quad (13.174)$$

Example 13.10 (Gauss) The characteristic function of the gaussian

$$P_G(x, \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \quad (13.175)$$

is by (3.18)

$$\begin{aligned} \tilde{P}_G(k, \mu, \sigma) &= \frac{1}{\sigma\sqrt{2\pi}} \int \exp\left(ikx - \frac{(x-\mu)^2}{2\sigma^2}\right) dx \quad (13.176) \\ &= \frac{e^{ik\mu}}{\sigma\sqrt{2\pi}} \int \exp\left(ikx - \frac{x^2}{2\sigma^2}\right) dx = \exp\left(i\mu k - \frac{1}{2}\sigma^2 k^2\right). \end{aligned}$$

□

For a discrete probability distribution P_n the characteristic function is

$$\chi(k) \equiv E[e^{ikx}] = \sum_n e^{ikx_n} \, P_n. \quad (13.177)$$

The normalization of both continuous and discrete probability distributions implies that their characteristic functions satisfy $\tilde{P}(0) = \chi(0) = 1$.

Example 13.11 (Poisson) The Poisson distribution (13.58)

$$P_P(n, \langle n \rangle) = \frac{\langle n \rangle^n}{n!} \, e^{-\langle n \rangle} \tag{13.178}$$

has the characteristic function

$$\chi(k) = \sum_{n=0}^{\infty} e^{ikn} \, \frac{\langle n \rangle^n}{n!} \, e^{-\langle n \rangle} = e^{-\langle n \rangle} \sum_{n=0}^{\infty} \frac{(\langle n \rangle e^{ik})^n}{n!} = \exp\left[\langle n \rangle \left(e^{ik} - 1\right)\right]. \tag{13.179}$$

□

The **moment-generating function** is the characteristic function evaluated at an imaginary argument

$$M(k) \equiv E[e^{kx}] = \tilde{P}(-ik) = \chi(-ik). \tag{13.180}$$

For a continuous probability distribution $P(x)$, it is

$$M(k) = E[e^{kx}] = \int e^{kx} \, P(x) \, dx \tag{13.181}$$

and for a discrete probability distribution P_n, it is

$$M(k) = E[e^{kx}] = \sum_n e^{kx_n} \, P_n. \tag{13.182}$$

In both cases, the normalization of the probability distribution implies that $M(0) = 1$.

Derivatives of the moment-generating function and of the characteristic function give the moments

$$E[x^n] = \mu_n = \left. \frac{d^n M(k)}{dk^n} \right|_{k=0} = (-i)^n \left. \frac{d^n \tilde{P}(k)}{dk^n} \right|_{k=0}. \tag{13.183}$$

Example 13.12 (Gauss and Poisson) The moment-generating functions for the distributions of Gauss (13.175) and Poisson (13.178) are

$$M_G(k, \mu, \sigma) = \exp\left(\mu k + \frac{1}{2}\sigma^2 k^2\right) \;\text{ and }\; M_P(k, \langle n \rangle) = \exp\left[\langle n \rangle \left(e^k - 1\right)\right]. \tag{13.184}$$

They give as the first three moments of these distributions

$$\mu_{G0} = 1, \quad \mu_{G1} = \mu, \quad \mu_{G2} = \mu^2 + \sigma^2 \tag{13.185}$$

$$\mu_{P0} = 1, \quad \mu_{P1} = \langle n \rangle, \quad \mu_{P2} = \langle n \rangle + \langle n \rangle^2 \tag{13.186}$$

(exercise 13.22). □

Since the characteristic and moment-generating functions have derivatives (13.183) proportional to the moments μ_n, their Taylor series are

$$\tilde{P}(k) = E[e^{ikx}] = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} E[x^n] = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \mu_n \tag{13.187}$$

and

$$M(k) = E[e^{kx}] = \sum_{n=0}^{\infty} \frac{k^n}{n!} E[x^n] = \sum_{n=0}^{\infty} \frac{k^n}{n!} \mu_n. \tag{13.188}$$

The **cumulants** c_n of a probability distribution are the derivatives of the logarithm of its moment-generating function

$$c_n = \left. \frac{d^n \ln M(k)}{dk^n} \right|_{k=0} = (-i)^n \left. \frac{d^n \ln \tilde{P}(k)}{dk^n} \right|_{k=0}. \tag{13.189}$$

One may show (exercise 13.24) that the first five cumulants of an arbitrary probability distribution are

$$c_0 = 0, \quad c_1 = \mu, \quad c_2 = \sigma^2, \quad c_3 = \nu_3, \quad \text{and} \quad c_4 = \nu_4 - 3\sigma^4 \tag{13.190}$$

where the ν's are its central moments (13.27). The 3d and 4th **normalized cumulants** are the **skewness** $\zeta = c_3/\sigma^3 = \nu_3/\sigma^3$ and the **kurtosis** $\kappa = c_4/\sigma^4 = \nu_4/\sigma^4 - 3$.

Example 13.13 (Gaussian Cumulants) The logarithm of the moment-generating function (13.184) of Gauss's distribution is $\mu k + \sigma^2 k^2/2$. Thus by (13.189), $P_G(x, \mu, \sigma)$ has no skewness or kurtosis, its cumulants vanish $c_{Gn} = 0$ for $n > 2$, and its fourth central moment is $\nu_4 = 3\sigma^4$. □

13.13 Fat Tails

The gaussian probability distribution $P_G(x, \mu, \sigma)$ falls off for $|x - \mu| \gg \sigma$ very fast—as $\exp\left(-(x-\mu)^2/2\sigma^2\right)$. Many other probability distributions fall off more slowly; they have **fat tails**. Rare "black-swan" events—wild fluctuations, market bubbles, and crashes—lurk in their fat tails.

Gosset's distribution, which is known as **Student's t-distribution** with ν degrees of freedom

$$P_S(x, \nu, a) = \frac{1}{\sqrt{\pi}} \frac{\Gamma((1+\nu)/2)}{\Gamma(\nu/2)} \frac{a^\nu}{(a^2 + x^2)^{(1+\nu)/2}} \tag{13.191}$$

has **power-law tails**. Its even moments are

$$\mu_{2n} = (2n-1)!!\,\frac{\Gamma(\nu/2-n)}{\Gamma(\nu/2)}\left(\frac{a^2}{2}\right)^n \tag{13.192}$$

for $2n < \nu$ and infinite otherwise. For $\nu = 1$, it coincides with the Breit-Wigner or Cauchy distribution

$$P_S(x,1,a) = \frac{1}{\pi}\,\frac{a}{a^2+x^2} \tag{13.193}$$

in which $x = E - E_0$ and $a = \Gamma/2$ is the half-width at half-maximum.

Two representative cumulative probabilities are (Bouchaud and Potters, 2003, p.15–16)

$$\Pr(x,\infty) = \int_x^\infty P_S(x',3,1)\,dx' = \frac{1}{2} - \frac{1}{\pi}\left[\arctan x + \frac{x}{1+x^2}\right] \tag{13.194}$$

$$\Pr(x,\infty) = \int_x^\infty P_S(x',4,\sqrt{2})\,dx' = \frac{1}{2} - \frac{3}{4}u + \frac{1}{4}u^3 \tag{13.195}$$

where $u = x/\sqrt{2+x^2}$ and a is picked so $\sigma^2 = 1$. William Gosset (1876–1937), who worked for Guinness, wrote as Student because Guinness didn't let its employees publish.

The **log-normal** probability distribution on $(0,\infty)$

$$P_{\ln}(x) = \frac{1}{\sigma x\sqrt{2\pi}}\exp\left[-\frac{\ln^2(x/x_0)}{2\sigma^2}\right] \tag{13.196}$$

describes distributions of rates of return (Bouchaud and Potters, 2003, p. 9). Its moments are (exercise 13.27)

$$\mu_n = x_0^n\, e^{n^2\sigma^2/2}. \tag{13.197}$$

The **exponential distribution** on $[0,\infty)$

$$P_e(x) = \alpha e^{-\alpha x} \tag{13.198}$$

has (exercise 13.28) mean $\mu = 1/\alpha$ and variance $\sigma^2 = 1/\alpha^2$. The sum of n independent exponentially and identically distributed random variables $x = x_1 + \cdots + x_n$ is distributed on $[0,\infty)$ as (Feller, 1966, p.10)

$$P_{n,e}(x) = \alpha\frac{(\alpha x)^{n-1}}{(n-1)!}e^{-\alpha x}. \tag{13.199}$$

The sum of the squares $x^2 = x_1^2 + \cdots + x_n^2$ of n independent normally and identically distributed random variables of zero mean and variance σ^2 gives rise to Pearson's **chi-squared distribution** on $(0, \infty)$

$$P_{n,G}(x,\sigma)dx = \frac{\sqrt{2}}{\sigma}\frac{1}{\Gamma(n/2)}\left(\frac{x}{\sigma\sqrt{2}}\right)^{n-1} e^{-x^2/(2\sigma^2)}dx \tag{13.200}$$

which for $x = v$, $n = 3$, and $\sigma^2 = kT/m$ is (exercise 13.29) the Maxwell-Boltzmann distribution (13.100). In terms of $\chi = x/\sigma$, it is

$$P_n(\chi^2/2)\, d\chi^2 = \frac{1}{\Gamma(n/2)}\left(\frac{\chi^2}{2}\right)^{n/2-1} e^{-\chi^2/2} d\left(\chi^2/2\right). \tag{13.201}$$

It has mean and variance

$$\mu = n \quad \text{and} \quad \sigma^2 = 2n \tag{13.202}$$

and is used in the chi-squared test (Pearson, 1900).

Personal income, the amplitudes of catastrophes, the price changes of financial assets, and many other phenomena occur on both small and large scales. **Lévy** distributions describe such multi-scale phenomena. The characteristic function for a symmetric Lévy distribution is for $\nu \le 2$

$$\tilde{L}_\nu(k, a_\nu) = \exp\left(-a_\nu |k|^\nu\right). \tag{13.203}$$

Its inverse Fourier transform (13.174) is for $\nu = 1$ (exercise 13.30) the **Cauchy** or **Lorentz** distribution

$$L_1(x, a_1) = \frac{a_1}{\pi(x^2 + a_1^2)} \tag{13.204}$$

and for $\nu = 2$ the gaussian

$$L_2(x, a_2) = P_G(x, 0, \sqrt{2a_2}) = \frac{1}{2\sqrt{\pi a_2}} \exp\left(-\frac{x^2}{4a_2}\right) \tag{13.205}$$

but for other values of ν no simple expression for $L_\nu(x, a_\nu)$ is available. For $0 < \nu < 2$ and as $x \to \pm\infty$, it falls off as $|x|^{-(1+\nu)}$, and for $\nu > 2$ it assumes negative values, ceasing to be a probability distribution (Bouchaud and Potters, 2003, pp. 10–13).

13.14 The Central Limit Theorem and Jarl Lindeberg

We have seen in sections (13.7 & 13.8) that unbiased fluctuations tend to distribute the position and velocity of molecules according to Gauss's distribution (13.75). Gaussian distributions occur very frequently. The **central limit theorem** suggests why they occur so often.