13.11 Fluctuation and Dissipation

Let’s look again at Langevin’s equation (13.116) but with $u$ as the independent variable

$$\frac{dv(u)}{du} + v(u) = a(u). \quad (13.146)$$

If we multiply both sides by the exponential $\exp(u/\tau)$

$$\left( \frac{dv}{du} + \frac{v}{\tau} \right) e^{u/\tau} = \frac{d}{du} \left( v e^{u/\tau} \right) = a(u) e^{u/\tau} \quad (13.147)$$

and integrate from 0 to $t$

$$\int_0^t \frac{d}{du} \left( v e^{u/\tau} \right) du = v(t) e^{t/\tau} - v(0) = \int_0^t a(u) e^{u/\tau} du \quad (13.148)$$

then we get

$$v(t) = e^{-t/\tau} v(0) + e^{-t/\tau} \int_0^t a(u) e^{u/\tau} du. \quad (13.149)$$

Thus the ensemble average of the square of the velocity is

$$\langle v^2(t) \rangle = e^{-2t/\tau} \langle v^2(0) \rangle + 2 e^{-2t/\tau} \int_0^t \langle v(t) \cdot a(u) \rangle e^{u/\tau} du \quad (13.150)$$

$$+ e^{-2t/\tau} \int_0^t \int_0^t \langle a(u_1) \cdot a(u_2) \rangle e^{(u_1+u_2)/\tau} du_1 du_2.$$

The second term on the RHS is zero, so we have

$$\langle v^2(t) \rangle = e^{-2t/\tau} \langle v^2(0) \rangle + e^{-2t/\tau} \int_0^t \int_0^t \langle a(u_1) \cdot a(u_2) \rangle e^{(u_1+u_2)/\tau} du_1 du_2. \quad (13.151)$$

The ensemble average

$$C(u_1, u_2) = \langle a(u_1) \cdot a(u_2) \rangle \quad (13.152)$$

is an example of an autocorrelation function.

All autocorrelation functions have some simple properties, which are easy to prove (Pathria, 1972, p. 458):

1. If the system is independent of time, then its autocorrelation function for any given variable $A(t)$ depends only upon the time delay $s$:

$$C(t, t + s) = \langle A(t) \cdot A(t + s) \rangle = \langle A(t) \rangle \quad (13.153)$$

2. The autocorrelation function for $s = 0$ is necessarily non-negative

$$C(t, t) = \langle A(t) \cdot A(t) \rangle = \langle A(t)^2 \rangle \geq 0. \quad (13.154)$$
If the system is time independent, then \( C(t, t) = C(0) \geq 0 \).

3. The absolute value of \( C(t_1, t_2) \) is never greater than the average of \( C(t_1, t_1) \) and \( C(t_2, t_2) \) because

\[
\langle |A(t_1) \pm A(t_2)|^2 \rangle = \langle A(t_1)^2 \rangle + \langle A(t_2)^2 \rangle \pm 2\langle A(t_1) \cdot A(t_2) \rangle \geq 0 \quad (13.155)
\]

which implies that

\[
-C(t_1, t_2) \leq \frac{1}{2} (C(t_1, t_1) + C(t_2, t_2)) \geq C(t_1, t_2) \quad (13.156)
\]

or

\[
|C(t_1, t_2)| \leq \frac{1}{2} (C(t_1, t_1) + C(t_2, t_2)). \quad (13.157)
\]

For a time-independent system, this inequality is \( |C(s)| \leq C(0) \) for every time delay \( s \).

4. If the variables \( A(t_1) \) and \( A(t_2) \) commute, then their autocorrelation function is symmetric

\[
C(t_1, t_2) = \langle A(t_1) \cdot A(t_2) \rangle = \langle A(t_2) \cdot A(t_1) \rangle = C(t_2, t_1). \quad (13.158)
\]

For a time-independent system, this symmetry is \( C(s) = C(-s) \).

5. If the variable \( A(t) \) is randomly fluctuating with zero mean, then we expect both that its ensemble average vanishes

\[
\langle A(t) \rangle = 0 \quad (13.159)
\]

and that there is some characteristic time scale \( T \) beyond which the correlation function falls to zero:

\[
\langle A(t_1) \cdot A(t_2) \rangle \to \langle A(t_1) \rangle \cdot \langle A(t_2) \rangle = 0 \quad (13.160)
\]

when \( |t_1 - t_2| \gg T \).

In terms of the autocorrelation function \( C(u_1, u_2) = \langle a(u_1) \cdot a(u_2) \rangle \) of the acceleration, the variance of the velocity (13.151) is

\[
\langle v^2(t) \rangle = e^{-2t/\tau} \langle v^2(0) \rangle + e^{-2t/\tau} \int_0^t \int_0^t C(u_1, u_2) e^{(u_1+u_2)/\tau} \, du_1 \, du_2. \quad (13.161)
\]

Since \( C(u_1, u_2) \) is big only for tiny values of \( |u_2 - u_1| \), it makes sense to change variables to

\[
s = u_2 - u_1 \quad \text{and} \quad w = \frac{1}{2} (u_1 + u_2). \quad (13.162)
\]

The element of area then is by (12.6–12.14)

\[
du_1 \wedge du_2 = dw \wedge ds \quad (13.163)
\]