The mean number of successes

$$\mu = \langle n \rangle_B = \sum_{n=0}^{N} n \, P_B(n, p, N) = \sum_{n=0}^{N} n \binom{N}{n} p^n q^{N-n} \tag{13.45}$$

is a partial derivative with respect to $p$ with $q$ held fixed

$$\begin{aligned} \langle n \rangle_B &= p \frac{\partial}{\partial p} \sum_{n=0}^{N} \binom{N}{n} p^n q^{N-n} \\ &= p \frac{\partial}{\partial p} (p+q)^N = Np \, (p+q)^{N-1} = Np \end{aligned} \tag{13.46}$$

which verifies the estimate (13.42).

One may show (exercise 13.9) that the variance (13.21) of the binomial distribution is

$$V_B = \langle (n - \langle n \rangle)^2 \rangle = p \, (1-p) \, N. \tag{13.47}$$

Its standard deviation (13.23) is

$$\sigma_B = \sqrt{V_B} = \sqrt{p \, (1-p) \, N}. \tag{13.48}$$

The ratio of the width to the mean

$$\frac{\sigma_B}{\langle n \rangle_B} = \frac{\sqrt{p \, (1-p) \, N}}{Np} = \sqrt{\frac{1-p}{Np}} \tag{13.49}$$

decreases with $N$ as $1/\sqrt{N}$.

**Example 13.5** (Avogadro's number) A mole of gas is Avogadro's number $N_A = 6 \times 10^{23}$ of molecules. If the gas is in a cubical box, then the chance that each molecule will be in the left half of the cube is $p = 1/2$. The mean number of molecules there is $\langle n \rangle_B = pN_A = 3 \times 10^{23}$, and the uncertainty in $n$ is $\sigma_B = \sqrt{p \, (1-p) \, N} = \sqrt{3 \times 10^{23}/4} = 3 \times 10^{11}$. So the numbers of gas molecules in the two halves of the box are equal to within $\sigma_B / \langle n \rangle_B = 10^{-12}$ or to 1 part in $10^{12}$. □

Because $N!$ increases very rapidly with $N$, the rule

$$P_B(n+1, p, N) = \frac{p}{1-p} \, \frac{N-n}{n+1} \, P_B(n, p, N) \tag{13.50}$$

is helpful when $N$ is big. But when $N$ exceeds a few hundred, the formula (13.43) for $P_B(n, p, N)$ becomes unmanageable even in quadruple precision.

One way of computing $P_B(n,p,N)$ for large $N$ is to use Srinivasa Ramanujan's correction (4.39) to Stirling's formula $N! \approx \sqrt{2\pi N}(N/e)^N$

$$N! \approx \sqrt{2\pi N}\left(\frac{N}{e}\right)^N \left(1+\frac{1}{2N}+\frac{1}{8N^2}\right)^{1/6}. \tag{13.51}$$

When $N$ and $N-n$, but not $n$, are big, one may use (13.51) for $N!$ and $(N-n)!$ in the formula (13.43) for $P_B(n,p,N)$ and so may show (exercise 13.11) that

$$P_B(n,p,N) \approx \frac{(pN)^n}{n!}\, q^{N-n}\, R_2(n,N) \tag{13.52}$$

in which

$$R_2(n,N) = \left(1-\frac{n}{N}\right)^{n-1/2}\left(1+\frac{1}{2N}+\frac{1}{8N^2}\right)^{1/6} \times\left[1+\frac{1}{2(N-n)}+\frac{1}{8(N-n)^2}\right]^{-1/6} \tag{13.53}$$

tends to unity as $N \to \infty$ for any fixed $n$.

When all three factorials in $P_B(n,p,N)$ are huge, one may use Ramanujan's approximation (13.51) to show (exercise 13.12) that

$$P_B(n,p,N) \approx \sqrt{\frac{N}{2\pi n(N-n)}}\left(\frac{pN}{n}\right)^n\left(\frac{qN}{N-n}\right)^{N-n} R_3(n,N) \tag{13.54}$$

where

$$R_3(n,N) = \left(1+\frac{1}{2n}+\frac{1}{8n^2}\right)^{-1/6}\left(1+\frac{1}{2N}+\frac{1}{8N^2}\right)^{1/6} \times\left[1+\frac{1}{2(N-n)}+\frac{1}{8(N-n)^2}\right]^{-1/6} \tag{13.55}$$

tends to unity as $N \to \infty$, $N-n \to \infty$, and $n \to \infty$.

Another way of coping with the unwieldy factorials in the binomial formula $P_B(n,p,N)$ is to use limiting forms of (13.43) due to Poisson and to Gauss.

## 13.4 The Poisson Distribution

Poisson took the two limits $N \to \infty$ and $p = \langle n\rangle/N \to 0$. So we let $N$ and $N-n$, but not $n$, tend to infinity, and use (13.52) for the binomial distribution (13.43). Since $R_2(n,N) \to 1$ as $N \to \infty$, we get

$$P_B(n,p,N) \approx \frac{(pN)^n}{n!}\, q^{N-n} = \frac{\langle n\rangle^n}{n!}\, q^{N-n}. \tag{13.56}$$