If we substitute our formula (13.169) for $\langle \boldsymbol{v}^2(t)\rangle$ into the expression (13.123) for the acceleration of $\langle \boldsymbol{r}^2\rangle$, then we get

$$\frac{d^2\langle \boldsymbol{r}^2(t)\rangle}{dt^2} = -\frac{1}{\tau}\frac{d\langle \boldsymbol{r}^2(t)\rangle}{dt} + 2e^{-2t/\tau}\langle \boldsymbol{v}^2(0)\rangle + \frac{6kT}{m}\left(1 - e^{-2t/\tau}\right). \quad (13.171)$$

The solution with both $\langle \boldsymbol{r}^2(0)\rangle = 0$ and $d\langle \boldsymbol{r}^2(0)\rangle/dt = 0$ is (exercise 13.21)

$$\langle \boldsymbol{r}^2(t)\rangle = \langle \boldsymbol{v}^2(0)\rangle\, \tau^2 \left(1 - e^{-t/\tau}\right)^2 - \frac{3kT}{m}\,\tau^2\left(1 - e^{-t/\tau}\right)\left(3 - e^{-t/\tau}\right) + \frac{6kT\tau}{m}\, t. \quad (13.172)$$

13.12 Characteristic and Moment-Generating Functions

The Fourier transform (3.9) of a probability distribution $P(x)$ is its **characteristic function** $\tilde{P}(k)$ sometimes written as $\chi(k)$

$$\tilde{P}(k) \equiv \chi(k) \equiv E[e^{ikx}] = \int e^{ikx}\, P(x)\, dx. \quad (13.173)$$

The probability distribution $P(x)$ is the inverse Fourier transform (3.9)

$$P(x) = \int e^{-ikx}\, \tilde{P}(k)\, \frac{dk}{2\pi}. \quad (13.174)$$

Example 13.10 (Gauss) The characteristic function of the gaussian

$$P_G(x, \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}}\, \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \quad (13.175)$$

is by (3.18)

$$\begin{aligned}\tilde{P}_G(k, \mu, \sigma) &= \frac{1}{\sigma\sqrt{2\pi}} \int \exp\left(ikx - \frac{(x-\mu)^2}{2\sigma^2}\right)\, dx \quad (13.176)\\ &= \frac{e^{ik\mu}}{\sigma\sqrt{2\pi}} \int \exp\left(ikx - \frac{x^2}{2\sigma^2}\right)\, dx = \exp\left(i\mu k - \frac{1}{2}\sigma^2 k^2\right).\end{aligned}$$

□

For a discrete probability distribution P_n the characteristic function is

$$\chi(k) \equiv E[e^{ikx}] = \sum_n e^{ikx_n}\, P_n. \quad (13.177)$$

The normalization of both continuous and discrete probability distributions implies that their characteristic functions satisfy $\tilde{P}(0) = \chi(0) = 1$.