$\Delta q_i$, one may show (exercise 12.10) that this sum of areas remains constant
\[
\frac{d}{dt} d\omega^1(\delta p, \delta q; \Delta p, \Delta q) = 0 \tag{12.77}
\]
along the trajectories in phase space (Gutzwiller, 1990, chap. 7).

**Example 12.12 (The Curl)** We saw in example 12.7 that the 1-form (12.50) of a vector field $\mathbf{A}$ is $\omega_A = A_1 \, dx_1 + A_2 \, dx_2 + A_3 \, dx_3$ in which the $h_k$'s are those that determine (12.44) the squared length $ds^2 = h_k^2 \, dx_k^2$ of the triply orthogonal coordinate system with unit vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$. So the exterior derivative of the 1-form $\omega_A$ is
\[
d\omega_A = \sum_{i,k=1}^3 \partial_k (A_i \, h_i) \, dx_k \wedge dx_i
\]
\[
= \left[ \frac{\partial (A_3 \, h_3)}{\partial x_2} - \frac{\partial (A_2 \, h_2)}{\partial x_3} \right] \, dx_2 \wedge dx_3 \\
+ \left[ \frac{\partial (A_2 \, h_2)}{\partial x_1} - \frac{\partial (A_1 \, h_1)}{\partial x_2} \right] \, dx_1 \wedge dx_2 \\
+ \left[ \frac{\partial (A_1 \, h_1)}{\partial x_3} - \frac{\partial (A_3 \, h_3)}{\partial x_1} \right] \, dx_3 \wedge dx_1 \equiv \omega_{\nabla \times \mathbf{A}}. \tag{12.78}
\]

Comparison with Eq. (12.52) shows that the curl of $\mathbf{A}$ is
\[
\nabla \times \mathbf{A} = \frac{1}{h_2 h_3} \left( \frac{\partial A_3 \, h_3}{\partial x_2} - \frac{\partial A_2 \, h_2}{\partial x_3} \right) \, dx_2 \wedge dx_3 \hat{e}_1 + \ldots
\]
\[
= \frac{1}{h_1 h_2 h_3} \begin{vmatrix}
    h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\
    \partial_1 & \partial_2 & \partial_3 \\
    A_1 h_1 & A_2 h_2 & A_3 h_3 \\
\end{vmatrix}
\]
\[
= \frac{1}{h_1 h_2 h_3} \sum_{i,j,k=1}^3 \epsilon_{ijk} h_i \hat{e}_i \frac{\partial (A_k h_k)}{\partial x_j} \tag{12.79}
\]
as we saw in (11.240). This formula gives our earlier expressions for the curl in cylindrical and spherical coordinates (11.241 & 11.242).

**Example 12.13 (The Divergence)** We have seen in equations (12.48, 12.49, & 12.52) that the 2-form $\omega_A(U, V) = A \cdot (U \times V)$ of the vector field $\mathbf{A} = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3$ is
\[
\omega^2_A = A_1 \, h_2 \, h_3 \, dx_2 \wedge dx_3 + A_2 \, h_3 \, h_1 \, dx_3 \wedge dx_1 + A_3 \, h_1 \, h_2 \, dx_1 \wedge dx_2. \tag{12.80}
\]