sets of basic differentials. Then by applying the formula (12.24) to the function $y_k(x)$, we get

$$dy_k = \sum_{j=1}^{n} \frac{\partial y_k(x)}{\partial x_j} \, dx_j \tag{12.32}$$

which is the familiar rule for changing variables.

The **most general differential 1-form** ω on the space $\mathbb{R}^n$ with coordinates $x_1 \dots x_n$ is a linear combination of the basic differentials dx_i with coefficients $a_i(x)$ that are smooth functions of $x = (x_1, \dots, x_n)$

$$\omega = a_1(x) \, dx_1 + \dots a_n(x) \, dx_n. \tag{12.33}$$

The **basic differential 2-forms** are $dx_i \wedge dx_k$ defined as

$$dx_i \wedge dx_k(A, B) = \begin{vmatrix} dx_i(A) & dx_k(A) \\ dx_i(B) & dx_k(B) \end{vmatrix} = \begin{vmatrix} A_i & A_k \\ B_i & B_k \end{vmatrix} = A_i B_k - A_k B_i. \tag{12.34}$$

So in particular

$$dx_i \wedge dx_i = 0. \tag{12.35}$$

The **basic differential k-forms** $dx_1 \wedge \cdots \wedge dx_k$ are defined as

$$dx_1 \wedge \dots dx_k(A_1, \dots A_k) = \begin{vmatrix} dx_1(A_1) & \dots & dx_k(A_1) \\ \vdots & \ddots & \vdots \\ dx_1(A_k) & \dots & dx_k(A_k) \end{vmatrix} = \begin{vmatrix} A_{11} & \dots & A_{1k} \\ \vdots & \ddots & \vdots \\ A_{k1} & \dots & A_{kk} \end{vmatrix}. \tag{12.36}$$

Example 12.4 ($dx_3 \wedge dr^2$) If $r^2 = x_1^2 + x_2^2 + x_3^2$, then dr^2 is

$$dr^2 = 2(x_1 dx_1 + x_2 \, dx_2 + x_3 \, dx_3) \tag{12.37}$$

and the differential 2-form $\omega = dx_3 \wedge dr^2$ is

$$\omega = dx_3 \wedge 2(x_1 dx_1 + x_2 \, dx_2 + x_3 \, dx_3) = 2x_1 dx_3 \wedge dx_1 + 2x_2 dx_3 \wedge dx_2 \tag{12.38}$$

since in view of (12.35) $dx_3 \wedge dx_3 = 0$. So the value of the 2-form ω on the vectors $A = (1, 2, 3)$ and $B = (2, 1, 1)$ at the point $x = (3, 0, 3)$ is

$$\omega(A, B) = 2x_1 dx_3 \wedge dx_1(A, B) = 6 \begin{vmatrix} dx_3(A) & dx_1(A) \\ dx_3(B) & dx_1(B) \end{vmatrix} = 6 \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = 30. \tag{12.39}$$

On the vectors, $C = (1, 0, 0)$ and $D = (0, 0, 1)$ at $x = (2, 3, 4)$, this 2-form has the value $\omega(C, D) = -4$. □