There are two ways of thinking about differential forms. The Russian literature views a manifold as embedded in $\mathbb{R}^n$ and so is somewhat more straightforward. We will discuss it first.

**The Russian Way:** Suppose $x(t)$ is a curve with $x(0) = x$ on some manifold $M$, and $f(x(t))$ is a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ that maps points $x(t)$ into numbers. Then the differential $df(\dot{x}(t))$ maps $\dot{x}(t)$ at $x$ into

$$df\left(\frac{d}{dt}x(t)\right) = \frac{d}{dt} f(x(t)) = \sum_{j=1}^n \dot{x}(t)_j \frac{\partial f(x(t))}{\partial x_j} = \dot{x}(t) \cdot \nabla f(x(t))$$

(12.18)

all at $t = 0$. As physicists, we think of $df$ as a number—the change in the function $f(x)$ when its argument $x$ is changed by $dx$. Russian mathematicians think of $df$ as a linear map of tangent vectors $\dot{x}$ at $x$ into numbers. Since this map is linear, we may multiply the definition (12.18) by $dt$ and arrive at the more familiar formula

$$dt df\left(\frac{d}{dt}x(t)\right) = df\left(dt \frac{d}{dt}x(t)\right) = df(dx(t)) = dx(t) \cdot \nabla f(x(t))$$

(12.19)

all at $t = 0$. So

$$df(dx) = dx \cdot \nabla f.$$  

(12.20)

is the physicist’s $df$.

Since the differential $df$ is a linear map of vectors $\dot{x}(0)$ into numbers, it is a 1-form; since it is defined on vectors like $\dot{x}(0)$, it is a **differential 1-form**. The term **differential 1-form** underscores the fact that the actual value of the differential $df$ depends upon the vector $\dot{x}(0)$ and the point $x = x(0)$. Mathematicians call the space of vectors $\dot{x}(0)$ at the point $x = x(0)$ the **tangent space** $TM_x$. They say $df$ is a smooth map of the **tangent bundle** $TM$, which is the union of the tangent spaces for all points $x$ in the manifold $M$, to the real line, so $df : TM \to \mathbb{R}$.

In the special case in which $f(x) = x_i(x) = x_i$, the differential $dx_i(\dot{x}(t))$ by (12.18) is

$$dx_i(\dot{x}(t)) = \sum_{j=1}^n \dot{x}_j(t) \frac{\partial x_i(x)}{\partial x_j} = \sum_{j=1}^n \dot{x}_j(t) \frac{\partial x_i}{\partial x_j} = \sum_{j=1}^n \dot{x}(t)_j \delta_{ij} = \dot{x}_i(t).$$  

(12.21)

These $dx_i$’s are the **basic differentials**. Using $A$ for the vector $\dot{x}(t)$, we find from our definition (12.18) that

$$dx_i(A) = \sum_{j=1}^n A_j \frac{\partial x_i}{\partial x_j} = \sum_{j=1}^n A_j \delta_{ij} = A_i$$  

(12.22)