**density**

$$\rho_c = \frac{3H^2}{8\pi G}. \tag{11.418}$$

The ratio of the energy density $\rho$ to the critical energy density is called $\Omega$

$$\Omega = \frac{\rho}{\rho_c} = \frac{8\pi G}{3H^2}\,\rho. \tag{11.419}$$

From (11.417), we see that $\Omega$ is

$$\Omega = 1 + \frac{k}{(aH)^2} = 1 + \frac{k}{\dot{a}^2}. \tag{11.420}$$

Thus $\Omega = 1$ both in a flat universe ($k = 0$) and as $aH \to \infty$. One use of inflation is to expand $a$ by $10^{26}$ so as to force $\Omega$ almost exactly to unity.

Something like inflation is needed because in a universe in which the energy density is due to matter and/or radiation, the present value of $\Omega$

$$\Omega_0 = 1.000 \pm 0.036 \tag{11.421}$$

is unlikely. To see why, we note that conservation of energy ensures that $a^3$ times the matter density $\rho_m$ is constant. Radiation red-shifts by $a$, so energy conservation implies that $a^4$ times the radiation density $\rho_r$ is constant. So with $n = 3$ for matter and 4 for radiation, $\rho\, a^n \equiv 3\, F^2/8\pi G$ is a constant. In terms of $F$ and $n$, Friedmann's first-order equation (11.415) is

$$\dot{a}^2 = \frac{8\pi G}{3}\,\rho\, a^2 - k = \frac{F^2}{a^{n-2}} - k \tag{11.422}$$

In the small-$a$ limit of the early Universe, we have

$$\dot{a} = F/a^{(n-2)/2} \qquad \text{or} \qquad a^{(n-2)/2} da = F\, dt \tag{11.423}$$

which we integrate to $a \sim t^{2/n}$ so that $\dot{a} \sim t^{2/n-1}$. Now (11.420) says that

$$|\Omega - 1| = \frac{1}{\dot{a}^2} \propto t^{2-4/n} = \begin{cases} t & \text{radiation} \\ t^{2/3} & \text{matter} \end{cases}. \tag{11.424}$$

Thus, $\Omega$ deviated from unity faster than $t^{2/3}$ during the early Universe. At this rate, the inequality $|\Omega_0 - 1| < 0.036$ could last 13.8 billion years only if $\Omega$ at $t = 1$ second had been unity to within six parts in $10^{14}$. The only *known* explanation for such early flatness is inflation.

Manipulating our relation (11.420) between $\Omega$ and $aH$, we see that

$$(aH)^2 = \frac{k}{\Omega - 1}. \tag{11.425}$$

So $\Omega > 1$ implies $k = 1$, and $\Omega < 1$ implies $k = -1$, and as $\Omega \to 1$ the product $aH \to \infty$, which is the essence of flatness since curvature vanishes as the scale factor $a \to \infty$. Imagine blowing up a balloon.

Staying for the moment with a universe without inflation and with an energy density composed of radiation and/or matter, we note that the first-order equation (11.422) in the form $\dot{a}^2 = F^2/a^{n-2} - k$ tells us that for a closed ($k = 1$) universe, in the limit $a \to \infty$ we'd have $\dot{a}^2 \to -1$ which is impossible. Thus a closed universe eventually collapses, which is incompatible with the flatness (11.425) implied by the present value $\Omega_0 = 1.000 \pm 0.036$.

The first-order equation Friedmann (11.415) says that $\rho\, a^2 \geq 3k/8\pi G$. So in a closed universe ($k = 1$), the energy density $\rho$ is positive and increases without limit as $a \to 0$ as in a collapse. In open ($k < 0$) and flat ($k = 0$) universes, the same Friedmann equation (11.415) in the form $\dot{a}^2 = 8\pi G\rho a^2/3 - k$ tells us that if $\rho$ is positive, then $\dot{a}^2 > 0$, which means that $\dot{a}$ never vanishes. Hubble told us that $\dot{a} > 0$ now. So if our universe is open or flat, then it always expands.

Due to the expansion of the universe, the wave-length of radiation grows with the scale factor $a(t)$. A photon emitted at time $t$ and scale factor $a(t)$ with wave-length $\lambda(t)$ will be seen now at time $t_0$ and scale factor $a(t_0)$ to have a longer wave-length $\lambda(t_0)$

$$\frac{\lambda(t_0)}{\lambda(t)} = \frac{a(t_0)}{a(t)} = z + 1 \tag{11.426}$$

in which the **redshift** $z$ is the ratio

$$z = \frac{\lambda(t_0) - \lambda(t)}{\lambda(t)} = \frac{\Delta\lambda}{\lambda}. \tag{11.427}$$

Now $H = \dot{a}/a = da/(adt)$ implies $dt = da/(aH)$, and $z = a_0/a - 1$ implies $dz = -a_0 da/a^2$, so we find

$$dt = -\frac{dz}{(1+z)H(z)} \tag{11.428}$$

which relates time intervals to redshift intervals. An on-line calculator is available for macroscopic intervals (Wright, 2006).

## 11.49 Model Cosmologies

The 0-component of the energy-momentum conservation law (11.375) is

$$\begin{aligned} 0 = (T^a{}_0)_{;a} &= \partial_a T^a{}_0 + \Gamma^a_{ac} T^c{}_0 - T^a{}_c \Gamma^c_{0a} \\ &= -\partial_0 T_{00} - \Gamma^a_{a0} T_{00} - g^{cc} T_{cc} \Gamma^c_{0c} \\ &= -\dot{\rho} - 3\frac{\dot{a}}{a}\rho - 3p\frac{\dot{a}}{a} = -\dot{\rho} - 3\,\frac{\dot{a}}{a}\,(\rho + p)\,. \end{aligned} \tag{11.429}$$

or

$$\frac{d\rho}{da} = -\frac{3}{a}\,(\rho + p)\,. \tag{11.430}$$

The energy density $\rho$ is composed of fractions $\rho_k$ each contributing its own partial pressure $p_k$ according to its own **equation of state**

$$p_k = w_k \rho_k \tag{11.431}$$

in which $w_k$ is a constant. In terms of these components, the energy-momentum conservation law (11.430) is

$$\sum_k \frac{d\rho_k}{da} = -\frac{3}{a}\sum_k (1 + w_k)\ \rho_k \tag{11.432}$$

with solution

$$\rho = \sum_k \overline{\rho_k}\left(\frac{\overline{a}}{a}\right)^{3(1+w_k)} = \sum_k \overline{\rho_k}\left(\frac{\overline{a}}{a}\right)^{3(1+\overline{p_k}/\overline{\rho_k})}. \tag{11.433}$$

Simple cosmological models take the energy density and pressure each to have a single component with $p = w\rho$, and in this case

$$\rho = \overline{\rho}\left(\frac{\overline{a}}{a}\right)^{3(1+w)} = \overline{\rho}\left(\frac{\overline{a}}{a}\right)^{3(1+\overline{p}/\overline{\rho})}. \tag{11.434}$$

**Example 11.25** ($w = -1/3$, No Acceleration) If $w = -1/3$, then $p = w\,\rho = -\rho/3$ and $\rho + 3p = 0$. The second-order Friedmann equation (11.413) then tells us that $\ddot{a} = 0$. The scale factor does not accelerate.

To find its constant speed, we use its equation of state (11.434)

$$\rho = \overline{\rho}\left(\frac{\overline{a}}{a}\right)^{3(1+w)} = \overline{\rho}\left(\frac{\overline{a}}{a}\right)^{2}. \tag{11.435}$$

Now all the terms in Friedmann's first-order equation (11.415) have a common factor of $1/a^2$ which cancels leaving us with the square of the constant speed

$$\dot{a}^2 = \frac{8\pi G}{3}\,\overline{\rho}\,\overline{a}^2 - k. \tag{11.436}$$

Incidentally, $\overline{\rho}\,\overline{a}^2$ must exceed $3k/8\pi G$. The scale factor grows linearly with time as

$$a(t) = \left(\frac{8\pi G}{3}\,\overline{\rho}\,\overline{a}^2 - k\right)^{1/2}(t - t_0) + a(t_0). \tag{11.437}$$

Setting $t_0 = 0$ and $a(0) = 0$, we use the definition of the Hubble parameter $H = \dot{a}/a$ to write the constant linear growth $\dot{a}$ as $aH$ and the time as

$$t = \int_0^a da'/a'H = (1/aH) \int_0^a da' = 1/H. \tag{11.438}$$

So in a universe without acceleration, the age of the universe is the inverse of the Hubble rate. For our universe, the present Hubble time is $1/H_0 =$ 14.5 billion years, which isn't far from the actual age of $13.817 \pm 0.048$ billion years. Presumably, a slower Hubble rate during the era of matter compensates for the higher rate during the era of dark energy. □

**Example 11.26** ($w = -1$, Inflation) Inflation occurs when the ground state of the theory has a positive and constant energy density $\rho > 0$ that dwarfs the energy densities of the matter and radiation. The **internal energy** of the universe then is proportional to its volume $U = \rho V$, and the pressure $p$ as given by the thermodynamic relation

$$p = -\frac{\partial U}{\partial V} = -\rho \tag{11.439}$$

is **negative**. The equation of state (11.431) tells us that in this case $w = -1$. The second-order Friedmann equation (11.413) becomes

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) = \frac{8\pi G\rho}{3} \equiv g^2 \tag{11.440}$$

By it and the first-order Friedmann equation (11.415) and by choosing $t = 0$ as the time at which the scale factor $a$ is minimal, one may show (exercise 11.37) that in a closed ($k = 1$) universe

$$a(t) = \frac{\cosh g\,t}{g}. \tag{11.441}$$

Similarly in an open ($k = -1$) universe with $a(0) = 0$, we have

$$a(t) = \frac{\sinh g\,t}{g}. \tag{11.442}$$

Finally in a flat ($k = 0$) expanding universe, the scale factor is

$$a(t) = a(0)\,\exp(g\,t). \tag{11.443}$$

Studies of the cosmic microwave background radiation suggest that inflation did occur in the **very early** universe—possibly on a time scale as short as $10^{-35}$ s. What is the origin of the vacuum energy density $\rho$ that drove inflation? Current theories attribute it to the assumption by at least one scalar field $\phi$ of a mean value $\langle\phi\rangle$ different from the one $\langle 0|\phi|0\rangle$ that minimizes the energy density of the vacuum. When $\langle\phi\rangle$ settled to $\langle 0|\phi|0\rangle$, the vacuum energy was released as radiation and matter in a **Big Bang**. □

**Example 11.27** ($w = 1/3$, The Era of Radiation) Until a redshift of $z = 3400$ or 50,000 years after inflation, our universe was dominated by **radiation** (Frieman et al., 2008). During *The First Three Minutes* (Weinberg, 1988) of the era of radiation, the quarks and gluons formed hadrons, which decayed into protons and neutrons. As the neutrons decayed ($\tau = 885.7$ s), they and the protons formed the light elements—principally hydrogen, deuterium, and helium in a process called **big-bang nucleosynthesis**.

We can guess the value of $w$ for radiation by noticing that the energy-momentum tensor of the electromagnetic field (in suitable units)

$$T^{ab} = F^a{}_c F^{bc} - \frac{1}{4} g^{ab} F_{cd} F^{cd} \tag{11.444}$$

is traceless

$$T = T^a{}_a = F^a{}_c F_a{}^c - \frac{1}{4} \delta^a_a F_{cd} F^{cd} = 0. \tag{11.445}$$

But by (11.412) its trace must be $T = 3p - \rho$. So for radiation $p = \rho/3$ and $w = 1/3$. The relation (11.434) between the energy density and the scale factor then is

$$\rho = \overline{\rho} \left( \frac{\overline{a}}{a} \right)^4 . \tag{11.446}$$

The energy drops both with the volume $a^3$ and with the scale factor $a$ due to a redshift; so it drops as $1/a^4$. Thus the quantity

$$f^2 \equiv \frac{8\pi G \rho a^4}{3} \tag{11.447}$$

is a constant. The Friedmann equations (11.413 & 11.414) now are

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3p) = -\frac{8\pi G \rho}{3} \quad \text{or} \quad \ddot{a} = -\frac{f^2}{a^3} \tag{11.448}$$

and

$$\dot{a}^2 + k = \frac{f^2}{a^2}. \tag{11.449}$$

With calendars chosen so that $a(0) = 0$, this last equation (11.449) tells us that for a flat universe ($k = 0$)

$$a(t) = (2f\,t)^{1/2} \tag{11.450}$$

while for a closed universe ($k = 1$)

$$a(t) = \sqrt{f^2 - (t - f)^2} \tag{11.451}$$

and for an open universe ($k = -1$)

$$a(t) = \sqrt{(t + f)^2 - f^2} \tag{11.452}$$

as we saw in (6.422). The scale factor (11.451) of a closed universe of radiation has a maximum $a = f$ at $t = f$ and falls back to zero at $t = 2f$. □

**Example 11.28** ($w = 0$, The Era of Matter) A universe composed only of **dust** or **non-relativistic collisionless matter** has no pressure. Thus $p = w\rho = 0$ with $\rho \neq 0$, and so $w = 0$. Conservation of energy (11.433), or equivalently (11.434), implies that the energy density falls with the volume

$$\rho = \overline{\rho}\left(\frac{\overline{a}}{a}\right)^3. \tag{11.453}$$

As the scale factor $a(t)$ increases, the matter energy density, which falls as $1/a^3$, eventually dominates the radiation energy density, which falls as $1/a^4$. This happened in our universe about 50,000 years after inflation at a temperature of $T = 9,400$ K or $kT = 0.81$ eV. Were baryons most of the matter, the era of radiation dominance would have lasted for a few hundred thousand years. But the kind of matter that we know about, which interacts with photons, is only about 15% of the total; the rest—an unknown substance called **dark matter**—shortened the era of radiation dominance by nearly 2 million years.

Since $\rho \propto 1/a^3$, the quantity

$$m^2 = \frac{4\pi G\rho a^3}{3} \tag{11.454}$$

is a constant. For a matter-dominated universe, the Friedmann equations (11.413 & 11.414) then are

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) = -\frac{4\pi G\rho}{3} \quad \text{or} \quad \ddot{a} = -\frac{m^2}{a^2} \tag{11.455}$$

and

$$\dot{a}^2 + k = 2m^2/a. \tag{11.456}$$

For a flat universe, $k = 0$, we get

$$a(t) = \left[\frac{3m}{\sqrt{2}}\, t\right]^{2/3}. \tag{11.457}$$

For a closed universe, $k = 1$, we use example 6.47 to integrate

$$\dot{a} = \sqrt{2m^2/a - 1} \tag{11.458}$$

to

$$t - t_0 = -\sqrt{a(2m^2 - a)} - m^2 \arcsin(1 - a/m^2). \tag{11.459}$$

With a suitable calendar and choice of $t_0$, one may parametrize this solution in terms of the **development angle** $\phi(t)$ as

$$\begin{aligned} a(t) &= m^2 \left[1 - \cos\phi(t)\right] \\ t &= m^2 \left[\phi(t) - \sin\phi(t)\right]. \end{aligned} \tag{11.460}$$

For an open universe, $k = -1$, we use example 6.48 to integrate

$$\dot{a} = \sqrt{2m^2/a + 1} \tag{11.461}$$

to

$$t - t_0 = \left[a(2m^2 + a)\right]^{1/2} - m^2 \ln\left\{2\left[a(2m^2 + a)\right]^{1/2} + 2a + 2m^2\right\}. \tag{11.462}$$

The conventional parametrization is

$$\begin{aligned} a(t) &= m^2 \left[\cosh\phi(t) - 1\right] \\ t &= m^2 \left[\sinh\phi(t) - \phi(t)\right]. \end{aligned} \tag{11.463}$$

**Transparency**: Some 380,000 years after inflation at a redshift of $z = 1090$, the universe had cooled to about $T = 3000$ K or $kT = 0.26$ eV—a temperature at which less than 1% of the hydrogen is ionized. Ordinary matter became a gas of neutral atoms rather than a plasma of ions and electrons, and the universe suddenly became **transparent** to light. Some scientists call this moment of last scattering or first transparency **recombination**. □

**Example 11.29** ($w = -1$, The Era of Dark Energy) About 10.3 billion years after inflation at a redshift of $z = 0.30$, the matter density falling as $1/a^3$ dropped below the very small but positive value of the energy density $\rho_v = (2.23 \text{ meV})^4$ of the vacuum. The present time is 13.817 billion years after inflation. So for the past 3 billion years, this constant energy density, called **dark energy**, has accelerated the expansion of the universe approximately as (11.442)

$$a(t) = a(t_m) \exp\left((t - t_m)\sqrt{8\pi G\rho_v/3}\right) \tag{11.464}$$

in which $t_m = 10.3 \times 10^9$ years. □

Observations and measurements on the largest scales indicate that the universe is flat: $k = 0$. So the evolution of the scale factor $a(t)$ is given by the $k = 0$ equations (11.443, 11.450, 11.457, & 11.464) for a flat universe. During the brief era of inflation, the scale factor $a(t)$ grew as (11.443)

$$a(t) = a(0) \exp\left(t\sqrt{8\pi G\rho_i/3}\right) \tag{11.465}$$

in which $\rho_i$ is the positive energy density that drove inflation.

During the 50,000-year era of radiation, $a(t)$ grew as $\sqrt{t}$ as in (11.450)

$$a(t) = \left(2\,(t - t_i)\,\sqrt{8\pi G\rho(t_r')a^4(t_r')/3}\right)^{1/2} + a(t_i) \tag{11.466}$$

where $t_i$ is the time at the end of inflation, and $t_r'$ is any time during the era of radiation. During this era, the energy of highly relativistic particles dominated the energy density, and $\rho a^4 \propto T^4 a^4$ was approximately constant, so that $T(t) \propto 1/a(t) \propto 1/\sqrt{t}$. When the temperature was in the range $10^{12} > T > 10^{10}\,\text{K}$ or $m_\mu c^2 > kT > m_e c^2$, where $m_\mu$ is the mass of the muon and $m_e$ that of the electron, the radiation was mostly electrons, positrons, photons, and neutrinos, and the relation between the time $t$ and the temperature $T$ was (Weinberg, 2010, ch. 3)

$$t = 0.994 \ \text{sec} \times \left[\frac{10^{10}\,\text{K}}{T}\right]^2 + \ \text{constant}. \tag{11.467}$$

By $10^9$ K, the positrons had annihilated with electrons, and the neutrinos fallen out of equilibrium. Between $10^9$ K and $10^6$ K, when the energy density of nonrelativistic particles became relevant, the time-temperature relation was (Weinberg, 2010, ch. 3)

$$t = 1.78 \ \text{sec} \times \left[\frac{10^{10}\,\text{K}}{T}\right]^2 + \ \text{constant}'. \tag{11.468}$$

During the 10.3 billion years of the matter era, $a(t)$ grew as (11.457)

$$a(t) = \left[(t - t_r)\sqrt{3\pi G\rho(t'_m)a(t'_m)} + a^{3/2}(t_r)\right]^{2/3} + a(t_r) \tag{11.469}$$

where $t_r$ is the time at the end of the radiation era, and $t'_m$ is any time in the matter era. By 380,000 years, the temperature had dropped to 3000 K, the universe had become transparent, and the CMBR had begun to travel freely.

Over the past 3 billion years of the era of vacuum dominance, $a(t)$ has been growing exponentially (11.464)

$$a(t) = a(t_m)\,\exp\left((t - t_m)\sqrt{8\pi G\rho_v/3}\right) \tag{11.470}$$

in which $t_m$ is the time at the end of the matter era, and $\rho_v$ is the density of dark energy, which while vastly less than the energy density $\rho_i$ that drove inflation, currently amounts to 68.5% of the total energy density.

## 11.50 Yang-Mills Theory

The gauge transformation of an **abelian** gauge theory like electrodynamics multiplies a *single* charged field by a space-time-dependent *phase factor* $\phi'(x) = \exp(iq\theta(x))\,\phi(x)$. Yang and Mills generalized this gauge transformation to one that multiplies a *vector* $\phi$ of matter fields by a space-time dependent *unitary matrix* $U(x)$

$$\phi'_a(x) = \sum_{b=1}^{n} U_{ab}(x)\,\phi_b(x) \quad \text{or} \quad \phi'(x) = U(x)\,\phi(x) \tag{11.471}$$

and showed how to make the action of the theory invariant under such **non-abelian** gauge transformations. (The fields $\phi$ are scalars for simplicity.)

Since the matrix $U$ is unitary, inner products like $\phi^\dagger(x)\,\phi(x)$ are automatically invariant

$$\left(\phi^\dagger(x)\,\phi(x)\right)' = \phi^\dagger(x)U^\dagger(x)U(x)\phi(x) = \phi^\dagger(x)\phi(x). \tag{11.472}$$

But inner products of derivatives $\partial^i\phi^\dagger\,\partial_i\phi$ are not invariant because the derivative acts on the matrix $U(x)$ as well as on the field $\phi(x)$.

Yang and Mills made derivatives $D_i\phi$ that transform like the fields $\phi$

$$(D_i\phi)' = U\,D_i\phi. \tag{11.473}$$

To do so, they introduced **gauge-field matrices** $A_i$ that play the role of