and then apply the exterior derivative

$$d * F = \tfrac{1}{4} d \left(F^{k\ell} \eta_{k\ell mn} dx^m \wedge dx^n \right) = \tfrac{1}{4} \partial_p \left(F^{k\ell} \eta_{k\ell mn} \right) dx^p \wedge dx^m \wedge dx^n .$$

To get back to a 1-form like $j = j_k \, dx^k$, we apply a second Hodge star

$$\begin{aligned} * d * F &= \tfrac{1}{4} \, \partial_p \left(F^{k\ell} \, \eta_{k\ell mn} \right) * \left(dx^p \wedge dx^m \wedge dx^n \right) \\ &= \tfrac{1}{4} \, \partial_p \left(F^{k\ell} \, \eta_{k\ell mn} \right) g^{ps} g^{mt} g^{nu} \eta_{stuv} \, dx^v \\ &= \tfrac{1}{4} \, \partial_p \left(\sqrt{g} \, F^{k\ell} \right) \epsilon_{k\ell mn} \, g^{ps} g^{mt} g^{nu} \sqrt{g} \, \epsilon_{stuv} \, dx^v \qquad (11.206) \\ &= \tfrac{1}{4} \, \partial_p \left(\sqrt{g} \, F^{k\ell} \right) \epsilon_{k\ell mn} \, g^{ps} g^{mt} g^{nu} g^{wv} \, \epsilon_{stuv} \sqrt{g} \, dx_w \\ &= \tfrac{1}{4} \, \partial_p \left(\sqrt{g} \, F^{k\ell} \right) \epsilon_{k\ell mn} \, \epsilon_{pmnw} \, \frac{\sqrt{g}}{\det g_{ij}} \, dx_w \\ &= \frac{s}{4\sqrt{g}} \, \partial_p \left(\sqrt{g} \, F^{k\ell} \right) \epsilon_{k\ell mn} \, \epsilon_{pmnw} \, dx_w \end{aligned}$$

in which we used the definition (1.184) of the determinant. Levi-Civita's 4-symbol obeys the identity (exercise 11.17)

$$\epsilon_{k\ell mn} \, \epsilon^{pwmn} = 2 \left(\delta^p_k \, \delta^w_\ell - \delta^w_k \, \delta^p_\ell \right) . \qquad (11.207)$$

Applying it to $* \, d * F$, we get

$$* \, d * F = \frac{s}{2\sqrt{g}} \partial_p \left(\sqrt{g} \, F^{k\ell} \right) \left(\delta^p_k \, \delta^w_\ell - \delta^w_k \, \delta^p_\ell \right) dx_w = - \frac{s}{\sqrt{g}} \, \partial_p \left(\sqrt{g} \, F^{kp} \right) \, dx_k .$$

In our space-time $s = -1$. Setting $* \, d * F$ equal to $j = j_k \, dx^k = j^k \, dx_k$ multiplied by the permeability μ_0 of the vacuum, we arrive at expressions for the microscopic inhomogeneous Maxwell equations in terms of both tensors and forms

$$\partial_p \left(\sqrt{g} \, F^{kp} \right) = \mu_0 \, \sqrt{g} \, j^k \quad \text{and} \quad * \, d * F = \mu_0 \, j . \qquad (11.208)$$

They and the homogeneous Bianchi identity (11.93, 11.114, & 11.247)

$$\epsilon^{ijk\ell} \, \partial_\ell F_{jk} = dF = d \, dA = 0 \qquad (11.209)$$

are invariant under general coordinate transformations. □