in which σ is the sign of the Jacobian $\det(\partial x/\partial x')$. Levi-Civita's tensor is a **pseudo-tensor** because it doesn't change sign under the parity transformation $x'^i = -x^i$.

We get η with upper indices by using the inverse g^{nm} of the metric tensor

$$\begin{aligned}\eta^{ijk} &= g^{it}\, g^{ju}\, g^{kv}\, \eta_{tuv} = g^{it}\, g^{ju}\, g^{kv}\, \sqrt{g}\, \epsilon_{tuv} = \sqrt{g}\, \epsilon_{ijk}\, \det(g^{mn}) \\ &= \sqrt{g}\, \epsilon_{ijk}/\det(g_{mn}) = s\epsilon_{ijk}/\sqrt{g} = s\epsilon^{ijk}/\sqrt{g}\end{aligned} \tag{11.184}$$

in which s is the sign of the determinant $\det g_{ij} = sg$.

Similarly in 4 dimensions, Levi-Civita's **symbol** $\epsilon_{ijk\ell} \equiv \epsilon^{ijk\ell}$ is totally antisymmetric with $\epsilon_{0123} = 1$ in all coordinate systems. No meaning attaches to whether the indices of the Levi-Civita symbol are up or down; some authors even use the notation $\epsilon(ijk\ell)$ or $\epsilon[ijk\ell]$ to emphasize this fact.

In 4 dimensions, the Levi-Civita pseudo-**tensor** is

$$\eta_{ijk\ell} = \sqrt{g}\, \epsilon_{ijk\ell}. \tag{11.185}$$

It transforms as

$$\begin{aligned}\eta'_{ijk\ell} &= \sqrt{g'}\, \epsilon_{ijk\ell} = \left|\det\left(\frac{\partial x}{\partial x'}\right)\right| \sqrt{g}\, \epsilon_{ijk\ell} = \sigma \det\left(\frac{\partial x}{\partial x'}\right) \sqrt{g}\, \epsilon_{ijk\ell} \\ &= \sigma\, \frac{\partial x^t}{\partial x'^i}\, \frac{\partial x^u}{\partial x'^j}\, \frac{\partial x^v}{\partial x'^k}\, \frac{\partial x^w}{\partial x'^\ell}\, \sqrt{g}\, \epsilon_{tuvw} = \sigma\, \frac{\partial x^t}{\partial x'^i}\, \frac{\partial x^u}{\partial x'^j}\, \frac{\partial x^v}{\partial x'^k}\, \frac{\partial x^w}{\partial x'^\ell}\, \eta_{tuvw}\end{aligned} \tag{11.186}$$

where σ is the sign of the Jacobian $\det(\partial x/\partial x')$.

Raising the indices on η with $\det g_{ij} = sg$ we have

$$\begin{aligned}\eta^{ijk\ell} &= g^{it}\, g^{ju}\, g^{kv}\, g^{\ell w}\, \eta_{tuvw} = g^{it}\, g^{ju}\, g^{kv}\, g^{\ell w}\, \sqrt{g}\, \epsilon_{tuvw} = \sqrt{g}\, \epsilon_{ijk\ell}\, \det(g^{mn}) \\ &= \sqrt{g}\, \epsilon_{ijk\ell}/\det(g_{mn}) = s\, \epsilon_{ijk\ell}/\sqrt{g} \equiv s\, \epsilon^{ijk\ell}/\sqrt{g}.\end{aligned} \tag{11.187}$$

In n dimensions, one may define Levi-Civita's symbol $\epsilon(i_1 \dots i_n)$ as totally antisymmetric with $\epsilon(1 \dots n) = 1$ and his tensor as $\eta_{i_1 \dots i_n} = \sqrt{g}\, \epsilon(i_1 \dots i_n)$.

11.26 The Hodge Star

In 3 cartesian coordinates, the Hodge dual turns 1-forms into 2-forms

$$*\, dx = dy \wedge dz \qquad *\, dy = dz \wedge dx \qquad *\, dz = dx \wedge dy \tag{11.188}$$

and 2-forms into 1-forms

$$*\,(dx \wedge dy) = dz \qquad *\,(dy \wedge dz) = dx \qquad *\,(dz \wedge dx) = dy. \tag{11.189}$$