The analog of $\boldsymbol{F} = m\,\boldsymbol{a}$ is

$$m\,\frac{d^2x^i}{d\tau^2} = m\,\frac{du^i}{d\tau} = \frac{dp^i}{d\tau} = f^i \tag{11.71}$$

in which $p^0 = E/c$, and f^i is a 4-vector force.

Example 11.6 (Time Dilation and Proper Time) In the frame of a laboratory, a particle of mass m with 4-momentum $p^i_{lab} = (E/c, p, 0, 0)$ travels a distance L in a time t for a 4-vector displacement of $x^i_{lab} = (ct, L, 0, 0)$. In its own rest frame, the particle's 4-momentum and 4-displacement are $p^i_{rest} = (mc, 0, 0, 0)$ and $x^i_{rest} = (c\tau, 0, 0, 0)$. Since the Minkowski inner product of two 4-vectors is Lorentz invariant, we have

$$\left(p^i x_i\right)_{rest} = \left(p^i x_i\right)_{lab} \quad \text{or} \quad Et - pL = mc^2\tau = mc^2 t\sqrt{1 - v^2/c^2} \tag{11.72}$$

so a massive particle's phase $\exp(-ip^i x_i/\hbar)$ is $\exp(imc^2\tau/\hbar)$. □

Example 11.7 ($p + \pi \to \Sigma + K$) What is the minimum energy that a beam of pions must have to produce a sigma hyperon and a kaon by striking a proton at rest? Conservation of the energy-momentum 4-vector gives $p_p + p_\pi = p_\Sigma + p_K$. We set $c = 1$ and use this equality in the invariant form $(p_p + p_\pi)^2 = (p_\Sigma + p_K)^2$. We compute $(p_p + p_\pi)^2$ in the the $p_p = (m_p, \mathbf{0})$ frame and set it equal to $(p_\Sigma + p_K)^2$ in the frame in which the spatial momenta of the Σ and the K cancel:

$$\begin{aligned}(p_p + p_\pi)^2 &= p_p^2 + p_\pi^2 + 2p_p \cdot p_\pi = -m_p^2 - m_\pi^2 - 2m_p E_\pi \\ &= (p_\Sigma + p_K)^2 = -\left(m_\Sigma + m_K\right)^2.\end{aligned} \tag{11.73}$$

Thus, since the relevant masses (in MeV) are $m_{\Sigma^+} = 1189.4$, $m_{K^+} = 493.7$, $m_p = 938.3$, and $m_{\pi^+} = 139.6$, the minimum total energy of the pion is

$$E_\pi = \frac{(m_\Sigma + m_K)^2 - m_p^2 - m_\pi^2}{2m_p} \approx 1030 \quad \text{MeV} \tag{11.74}$$

of which 890 MeV is kinetic. □