The coefficients $e'_i \cdot e_j$ form an orthogonal matrix, and the linear operator

$$\sum_{i=1}^{n} e_i e_i'^{\mathsf{T}} = \sum_{i=1}^{n} |e_i\rangle\langle e_i'| \tag{11.21}$$

is an orthogonal (real, unitary) transformation. The change $x \to x'$ is a rotation plus a possible reflection (exercise 11.2).

Example 11.2 (A Euclidean Space of Two Dimensions) In two-dimensional euclidean space, one can describe the same point by euclidean (x, y) and polar (r, θ) coordinates. The derivatives

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \frac{\partial x}{\partial r} \quad \text{and} \quad \frac{\partial r}{\partial y} = \frac{y}{r} = \frac{\partial y}{\partial r} \tag{11.22}$$

respect the symmetry (11.18), but (exercise 11.1) these derivatives

$$\frac{\partial \theta}{\partial x} = -\frac{y}{r^2} \neq \frac{\partial x}{\partial \theta} = -y \quad \text{and} \quad \frac{\partial \theta}{\partial y} = \frac{x}{r^2} \neq \frac{\partial y}{\partial \theta} = x \tag{11.23}$$

do not. □

11.6 Summation Conventions

When a given index is repeated in a product, that index usually is being summed over. So to avoid distracting summation symbols, one writes

$$A_i\, B_i \equiv \sum_{i=1}^{n} A_i\, B_i. \tag{11.24}$$

The sum is understood to be over the relevant range of indices, usually from 0 or 1 to 3 or n. Where the distinction between covariant and contravariant indices matters, an index that appears twice in the same monomial, once as a subscript and once as a superscript, is a dummy index that is summed over as in

$$A_i\, B^i \equiv \sum_{i=1}^{n} A_i\, B^i. \tag{11.25}$$

These summation conventions make tensor notation almost as compact as matrix notation. They make equations easier to read and write.