relations (10.235–10.237), then so do

\[ J \text{ and } -K. \]  

(10.238)

The infinitesimal Lorentz transformation (10.234) is the 4 \( \times \) 4 matrix

\[
L = I + \omega = I + \theta t R_{\ell} + \lambda_j B_j = \begin{pmatrix}
1 & \lambda_1 & \lambda_2 & \lambda_3 \\
\lambda_1 & 1 & -\theta_3 & \theta_2 \\
\lambda_2 & \theta_3 & 1 & -\theta_1 \\
\lambda_3 & -\theta_2 & \theta_1 & 1
\end{pmatrix}. 
\]  

(10.239)

It moves any 4-vector \( x \) to \( x' = Lx \) or in components \( x'^a = L_a^b x^b \)

\[
x'^0 = x^0 + \lambda_1 x^1 + \lambda_2 x^2 + \lambda_3 x^3 \\
x'^1 = \lambda_1 x^0 + x^1 - \theta_3 x^2 + \theta_2 x^3 \\
x'^2 = \lambda_2 x^0 + \theta_3 x^1 + x^2 - \theta_1 x^3 \\
x'^3 = \lambda_3 x^0 - \theta_2 x^1 + \theta_1 x^2 + x^3.
\]  

(10.240)

More succinctly with \( t = x^0 \), this is

\[
t' = t + \lambda \cdot x \\
x' = x + t \lambda + \theta \wedge x
\]  

(10.241)
in which \( \wedge, \equiv, \times \) means cross-product.

For arbitrary real \( \theta \) and \( \lambda \), the matrices

\[ L = e^{-i\theta t J_\ell - i\lambda_k K_k} \]  

(10.242)
form the subgroup of \( SO(3,1) \) that is connected to the identity matrix \( I \).

This subgroup preserves the sign of the time of any time-like vector, that is, if \( x^2 < 0 \), and \( y = Lx \), then \( y^0 x^0 > 0 \). It is called the proper orthochronous Lorentz group. The rest of the (homogeneous) Lorentz group can be obtained from it by space \( P \), time \( T \), and space-time \( PT \) reflections.

The task of finding all the finite-dimensional irreducible representations of the proper orthochronous homogeneous Lorentz group becomes vastly simpler when we write the commutation relations (10.235–10.237) in terms of the hermitian matrices

\[ J_\ell^\pm = \frac{1}{2} (J_\ell \pm iK_\ell) \]  

(10.243)
which generate two independent rotation groups

\[
[J_\ell^+, J_\ell^+] = i\epsilon_{ijk} J_\ell^k \\
[J_\ell^-, J_\ell^-] = i\epsilon_{ijk} J_\ell^k \\
[J_\ell^+, J_\ell^-] = 0.
\]  

(10.244)
Thus the Lie algebra of the Lorentz group is equivalent to two copies of the Lie algebra (10.100) of $SU(2)$. Its finite-dimensional irreducible representations are the direct products

$$D^{(j,j')}(\theta, \lambda) = e^{-i\theta \lambda J_\ell - i\lambda \ell K_\ell} = e^{(-i\theta - \lambda \ell)J_\ell^+} e^{(-i\lambda + \ell \lambda)J_\ell^-}$$

(10.245)

of the nonunitary representations $D^{(j,0)}(\theta, \lambda) = e^{-i\theta \lambda J_\ell}$ and $D^{(0,j')}((\theta, \lambda) = e^{(-i\lambda + \ell \lambda)J_\ell^-}$ generated by the three $(2j + 1) \times (2j + 1)$ matrices $J_\ell^+$ and by the three $(2j' + 1) \times (2j' + 1)$ matrices $J_\ell^-$. Under a Lorentz transformation $L$, a field $\psi^{(j,j')}(x)$ that transforms under the $D^{(j,j')}$ representation of the Lorentz group responds as

$$U(L) \psi^{(j,j')}_{m,m'}(x) U^{-1}(L) = D_{mm'}^{(j,0)}(L^{-1}) D_{m'm''}^{(0,j')} (L^{-1}) \psi^{(j,j')}_{m''m}(Lx).$$

(10.246)

Although these representations are not unitary, the $SO(3)$ subgroup of the Lorentz group is represented unitarily by the hermitian matrices

$$J = J^+ + J^-.$$ 

(10.247)

Thus, the representation $D^{(j,j')}$ describes objects of the spins $s$ that can arise from the direct product of spin-$j$ with spin-$j'$ (Weinberg, 1995, p. 231)

$$s = j + j', j + j' - 1, \ldots, |j - j'|.$$ 

(10.248)

For instance, $D^{(0,0)}$ describes a spinless field or particle, while $D^{(1/2,0)}$ and $D^{(0,1/2)}$ respectively describe left-handed and right-handed spin-1/2 fields or particles. The representation $D^{(1/2,1/2)}$ describes objects of spin 1 and spin 0—the spatial and time components of a 4-vector.

The generators $K_j$ of the Lorentz boosts are related to $J^\pm$ by

$$K = -iJ^+ + iJ^-$$ 

(10.249)

which like (10.247) follows from the definition (10.243).

The interchange of $J^+$ and $J^-$ replaces the generators $J$ and $K$ with $J$ and $-K$, a substitution that we know (10.238) is legitimate.

10.32 Two-Dimensional Representations of the Lorentz Group

The generators of the representation $D^{(1/2,0)}$ with $j = 1/2$ and $j' = 0$ are given by (10.247 & 10.249) with $J^+ = \sigma/2$ and $J^- = 0$. They are

$$J = \frac{1}{2} \sigma \quad \text{and} \quad K = -i \frac{1}{2} \sigma.$$ 

(10.250)