under multiplication by an arbitrary complex number. The squared norm of q is its determinant

$$\|q\|^2 = |z|^2 + |w|^2 = \det q. \tag{10.176}$$

The matrix products $q^\dagger q$ and $q\,q^\dagger$ are the squared norm $\|q\|^2$ multiplied by the 2×2 identity matrix

$$q^\dagger q = q\,q^\dagger = \|q\|^2\, I \tag{10.177}$$

The 2×2 matrix

$$i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{10.178}$$

provides another expression for $\|q\|^2$ in terms of q and its transpose q^{T}

$$q^{\mathsf{T}} i\sigma_2\, q = \|q\|^2\, i\sigma_2. \tag{10.179}$$

Clearly $\|q\| = 0$ implies $q = 0$. The norm of a product of quaternions is the product of their norms

$$\|q_1 q_2\| = \sqrt{\det(q_1 q_2)} = \sqrt{\det q_1 \det q_2} = \|q_1\|\|q_2\|. \tag{10.180}$$

The quaternions therefore form an **associative division algebra** (over the real numbers); the only others are the real numbers and the complex numbers; the **octonions** are a nonassociative division algebra.

One may use the Pauli matrices to define for any real 4-vector x a **quaternion** $q(x)$ as

$$\begin{aligned} q(x) &= x_0 + i\,\sigma_k\, x_k = x_0 + i\,\boldsymbol{\sigma}\cdot\boldsymbol{x} \\ &= \begin{pmatrix} x_0 + i\,x_3 & x_2 + i\,x_1 \\ -\,x_2 + i\,x_1 & x_0 - i\,x_3 \end{pmatrix} \end{aligned} \tag{10.181}$$

with squared norm

$$\|q(x)\|^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2. \tag{10.182}$$

The product rule (10.116) for the Pauli matrices tells us that the product of two quaternions is

$$\begin{aligned} q(x)\,q(y) = &\ (x_0 + i\,\boldsymbol{\sigma}\cdot\boldsymbol{x})(y_0 + i\,\boldsymbol{\sigma}\cdot\boldsymbol{y}) \\ = &\ x_0 y_0 + i\,\boldsymbol{\sigma}\cdot(y_0\,\boldsymbol{x} + x_0\,\boldsymbol{y}) - \,i\,(\boldsymbol{x}\times\boldsymbol{y})\cdot\boldsymbol{\sigma} - \boldsymbol{x}\cdot\boldsymbol{y} \end{aligned} \tag{10.183}$$

so their commutator is

$$[q(x), q(y)] = \,-\,2\,i\,(\boldsymbol{x}\times\boldsymbol{y})\cdot\boldsymbol{\sigma}. \tag{10.184}$$