The trace formula (10.65) gives us the SU(3) structure constants as
\[ f_{abc} = -2i \text{Tr} ([t_a, t_b] t_c). \] (10.166)
They are real and totally antisymmetric with \( f_{123} = 1 \), \( f_{458} = f_{678} = \sqrt{3}/2 \), and \( f_{147} = f_{156} = f_{246} = f_{257} = f_{345} = -f_{367} = 1/2 \).

While no two generators of SU(2) commute, two generators of SU(3) do. In the representation (10.162,10.163), \( t_3 \) and \( t_8 \) are diagonal and so commute
\[ [t_3, t_8] = 0. \] (10.167)
They generate the Cartan subalgebra (section 10.26) of SU(3).

10.25 SU(3) and quarks

The generators defined by Eqs.(10.163 & 10.162) give us the \( 3 \times 3 \) representation
\[ D(\alpha) = \exp(i\alpha_a t_a) \] (10.168)
in which the sum \( a = 1, 2, \ldots 8 \) is over the eight generators \( t_a \). This representation acts on complex 3-vectors and is called the 3.

Note that if
\[ D(\alpha_1)D(\alpha_2) = D(\alpha_3) \] (10.169)
then the complex conjugates of these matrices obey the same multiplication rule
\[ D^*(\alpha_1)D^*(\alpha_2) = D^*(\alpha_3) \] (10.170)
and so form another representation of SU(3). It turns out that (unlike in SU(2)) this representation is inequivalent to the 3; it is the \( \overline{3} \).

There are three quarks with masses less than about 100 MeV/c\(^2\)—the u, d, and s quarks. The other three quarks c, b, and t are more massive by factors of 12, 45, and 173. Nobody knows why. Gell-Mann and Zweig suggested that the low-energy strong interactions were approximately invariant under unitary transformations of the three light quarks, which they represented by a 3, and of the three light antiquarks, which they represented by a \( \overline{3} \). They imagined that the eight light pseudo-scalar mesons, that is, the three pions \( \pi^- \), \( \pi^0 \), \( \pi^+ \), the neutral \( \eta \), and the four kaons \( K^0, K^+, K^- \overline{K^0} \), were composed of a quark and an antiquark. So they should transform as the tensor product
\[ 3 \otimes \overline{3} = 8 \oplus 1. \] (10.171)
They put the eight pseudo-scalar mesons into an 8.

They imagined that the eight light baryons — the two nucleons \(N\) and \(P\), the three sigmas \(\Sigma^-\), \(\Sigma^0\), \(\Sigma^+\), the neutral lambda \(\Lambda\), and the two cascades \(\Xi^-\) and \(\Xi^0\) — were each made of three quarks. They should transform as the tensor product

\[
3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1.
\]

They put the eight light baryons into one of these 8’s. When they were writing these papers, there were nine spin-3/2 resonances with masses somewhat heavier than 1200 MeV/c\(^2\) — four \(\Delta\)’s, three \(\Sigma^*\)’s, and two \(\Xi^*\)’s. They put these into the 10 and predicted the tenth and its mass. In 1964, a tenth spin-3/2 resonance, the \(\Omega^-\), was found with a mass close to their prediction of 1680 MeV/c\(^2\), and by 1973 an MIT-SLAC team had discovered quarks inside protons and neutrons. (George Zweig, 1937–)

### 10.26 Cartan Subalgebra

In any Lie group, the maximum set of mutually commuting generators \(H_a\) generate the **Cartan subalgebra**

\[
[H_a, H_b] = 0
\]

which is an abelian subalgebra. The number of generators in the Cartan subalgebra is the **rank** of the Lie algebra. The Cartan generators \(H_a\) can be simultaneously diagonalized, and their eigenvalues or diagonal elements are the **weights**

\[
H_a|\mu, x, D\rangle = \mu_a|\mu, x, D\rangle
\]

in which \(D\) labels the representation and \(x\) whatever other variables are needed to specify the state. The vector \(\mu\) is the **weight vector**. The **roots** are the weights of the adjoint representation.

### 10.27 Quaternions

If \(z\) and \(w\) are any two complex numbers, then the 2 \(\times\) 2 matrix

\[
q = \begin{pmatrix}
z & w \\
-w^* & z^*
\end{pmatrix}
\]

is a quaternion. The quaternions are closed under addition and multiplication and under multiplication by a real number (exercise 10.21), but not