relations (10.235–10.237), then so do

$$\boldsymbol{J} \quad \text{and} \quad -\boldsymbol{K}. \tag{10.238}$$

The infinitesimal Lorentz transformation (10.234) is the 4×4 matrix

$$L = I + \omega = I + \theta_\ell R_\ell + \lambda_j B_j = \begin{pmatrix} 1 & \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1 & 1 & -\theta_3 & \theta_2 \\ \lambda_2 & \theta_3 & 1 & -\theta_1 \\ \lambda_3 & -\theta_2 & \theta_1 & 1 \end{pmatrix}. \tag{10.239}$$

It moves any 4-vector x to $x' = L\,x$ or in components $x'^a = L^a{}_b\, x^b$

$$\begin{aligned} x'^0 &= x^0 + \lambda_1 x^1 + \lambda_2 x^2 + \lambda_3 x^3 \\ x'^1 &= \lambda_1 x^0 + x^1 - \theta_3 x^2 + \theta_2 x^3 \\ x'^2 &= \lambda_2 x^0 + \theta_3 x^1 + x^2 - \theta_1 x^3 \\ x'^3 &= \lambda_3 x^0 - \theta_2 x^1 + \theta_1 x^2 + x^3. \end{aligned} \tag{10.240}$$

More succinctly with $t = x^0$, this is

$$\begin{aligned} t' &= t + \boldsymbol{\lambda} \cdot \boldsymbol{x} \\ \boldsymbol{x}' &= \boldsymbol{x} + t\boldsymbol{\lambda} + \boldsymbol{\theta} \wedge \boldsymbol{x} \end{aligned} \tag{10.241}$$

in which $\wedge \equiv \times$ means cross-product.

For arbitrary real $\boldsymbol{\theta}$ and $\boldsymbol{\lambda}$, the matrices

$$L = e^{-i\theta_\ell J_\ell - i\lambda_\ell K_\ell} \tag{10.242}$$

form the subgroup of $SO(3,1)$ that is connected to the identity matrix I. This subgroup preserves the sign of the time of any time-like vector, that is, if $x^2 < 0$, and $y = Lx$, then $y^0 x^0 > 0$. It is called the proper orthochronous Lorentz group. The rest of the (homogeneous) Lorentz group can be obtained from it by space $\mathcal{P}$, time $\mathcal{T}$, and space-time $\mathcal{PT}$ reflections.

The task of finding all the finite-dimensional irreducible representations of the proper orthochronous homogeneous Lorentz group becomes vastly simpler when we write the commutation relations (10.235–10.237) in terms of the hermitian matrices

$$J^\pm_\ell = \frac{1}{2}\left(J_\ell \pm i K_\ell\right) \tag{10.243}$$

which generate two independent rotation groups

$$\begin{aligned} [J^+_i, J^+_j] &= i\epsilon_{ijk} J^+_k \\ [J^-_i, J^-_j] &= i\epsilon_{ijk} J^-_k \\ [J^+_i, J^-_j] &= 0. \end{aligned} \tag{10.244}$$

Thus the Lie algebra of the Lorentz group is equivalent to two copies of the Lie algebra (10.100) of $SU(2)$. Its finite-dimensional irreducible representations are the direct products

$$D^{(j,j')}(\boldsymbol{\theta}, \boldsymbol{\lambda}) = e^{-i\theta_\ell J_\ell - i\lambda_\ell K_\ell} = e^{(-i\theta_\ell - \lambda_\ell)J^+_\ell}\, e^{(-i\theta_\ell + \lambda_\ell)J^-_\ell} \tag{10.245}$$

of the nonunitary representations $D^{(j,0)}(\boldsymbol{\theta}, \boldsymbol{\lambda}) = e^{(-i\theta_\ell - \lambda_\ell)J^+_\ell}$ and $D^{(0,j')}(\boldsymbol{\theta}, \boldsymbol{\lambda}) = e^{(-i\theta_\ell + \lambda_\ell)J^-_\ell}$ generated by the three $(2j+1) \times (2j+1)$ matrices J^+_ℓ and by the three $(2j'+1) \times (2j'+1)$ matrices J^-_ℓ. Under a Lorentz transformation L, a field $\psi^{(j,j')}_{m,m'}(x)$ that transforms under the $D^{(j,j')}$ representation of the Lorentz group responds as

$$U(L)\, \psi^{(j,j')}_{m,m'}(x)\, U^{-1}(L) = D^{(j,0))}_{mm''}(L^{-1})\, D^{(0,j')}_{m'm'''}(L^{-1})\, \psi^{(j,j')}_{m'',m'''}(Lx). \tag{10.246}$$

Although these representations are not unitary, the $SO(3)$ subgroup of the Lorentz group is represented unitarily by the hermitian matrices

$$\boldsymbol{J} = \boldsymbol{J}^+ + \boldsymbol{J}^-. \tag{10.247}$$

Thus, the representation $D^{(j,j')}$ describes objects of the spins s that can arise from the direct product of spin-j with spin-j' (Weinberg, 1995, p. 231)

$$s = j + j',\, j + j' - 1, \ldots, |j - j'|. \tag{10.248}$$

For instance, $D^{(0,0)}$ describes a spinless field or particle, while $D^{(1/2,0)}$ and $D^{(0,1/2)}$ respectively describe left-handed and right-handed spin-1/2 fields or particles. The representation $D^{(1/2,1/2)}$ describes objects of spin 1 and spin 0—the spatial and time components of a 4-vector.

The generators K_j of the Lorentz boosts are related to $\boldsymbol{J}^\pm$ by

$$\boldsymbol{K} = -i\boldsymbol{J}^+ + i\boldsymbol{J}^- \tag{10.249}$$

which like (10.247) follows from the definition (10.243).

The interchange of $\boldsymbol{J}^+$ and $\boldsymbol{J}^-$ replaces the generators $\boldsymbol{J}$ and $\boldsymbol{K}$ with $\boldsymbol{J}$ and $-\boldsymbol{K}$, a substitution that we know (10.238) is legitimate.

10.32 Two-Dimensional Representations of the Lorentz Group

The generators of the representation $D^{(1/2,0)}$ with $j = 1/2$ and $j' = 0$ are given by (10.247 & 10.249) with $\boldsymbol{J}^+ = \boldsymbol{\sigma}/2$ and $\boldsymbol{J}^- = 0$. They are

$$\boldsymbol{J} = \frac{1}{2}\boldsymbol{\sigma} \quad \text{and} \quad \boldsymbol{K} = -i\frac{1}{2}\boldsymbol{\sigma}. \tag{10.250}$$