10.10 Finite Groups

A **finite group** is one that has a finite number of elements. The number of elements in a group is the **order** of the group.

Example 10.12 (Z_2) The group $\boldsymbol{Z_2}$ consists of two elements e and p with multiplication rules

$$ee = e, \quad ep = pe = p, \quad \text{and} \quad pp = e. \tag{10.35}$$

Clearly, Z_2 is abelian, and its order is 2. The identification $e \to 1$ and $p \to -1$ gives a 1-dimensional representation of the group Z_2 in terms of 1×1 matrices, which are just numbers. □

It is tedious to write the multiplication rules as individual equations. Normally people compress them into a multiplication table like this:

$\times$	e	p
e	e	p
p	p	e

(10.36)

A simple generalization of Z_2 is the group $\boldsymbol{Z_n}$ whose elements may be represented as $\exp(i2\pi m/n)$ for $m = 1, \ldots, n$. This group is also abelian, and its order is n.

Example 10.13 (Z_3) The multiplication table for $\boldsymbol{Z_3}$ is

$\times$	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

(10.37)

which says that $a^2 = b$, $b^2 = a$, and $ab = ba = e$. □

10.11 The Regular Representation

For any finite group G we can associate an orthonormal vector $|g_i\rangle$ with each element g_i of the group. So $\langle g_i|g_j\rangle = \delta_{ij}$. These orthonormal vectors $|g_i\rangle$ form a basis for a vector space whose dimension is the order of the group. The matrix $D(g_k)$ of the regular representation of G is defined to map any vector $|g_i\rangle$ into the vector $|g_k g_i\rangle$ associated with the product $g_k g_i$

$$D(g_k)|g_i\rangle = |g_k g_i\rangle. \tag{10.38}$$