of the representation D_2, and so D_2 would be reducible, which is contrary to our assumption that D_1 and D_2 are irreducible. So the null space $\mathcal{N}(A)$ must be the whole space upon which A acts, that is, $A = 0$.

A similar argument shows that if $\langle y|A = 0$ for some bra $\langle y|$, then $A = 0$.

So either A is zero or it annihilates no ket and no bra. In the latter case, A must be square and invertible, which would imply that $D_2(g) = A^{-1}D_1(g)A$, that is, that D_1 and D_2 are equivalent representations, which is contrary to our assumption that they are inequivalent. The only way out is that A vanishes.

Part 2: If for a finite-dimensional, irreducible representation $D(g)$ of a group G, we have $D(g)A = AD(g)$ for some matrix A and for all $g \in G$, then $A = cI$. That is, any matrix that commutes with every element of a finite-dimensional, irreducible representation must be a multiple of the identity matrix.

Proof: Every square matrix A has at least one eigenvector $|x\rangle$ and eigenvalue c so that $A|x\rangle = c|x\rangle$ because its characteristic equation $\det(A-cI) = 0$ always has at least one root by the fundamental theorem of algebra (5.73). So the null space $\mathcal{N}(A - cI)$ has dimension greater than zero. The assumption $D(g)A = AD(g)$ for all $g \in G$ implies that $D(g)(A-cI) = (A-cI)D(g)$ for all $g \in G$. Let P be the projection operator onto the null space $\mathcal{N}(A-cI)$. Then we have $(A - cI)D(g)P = D(g)(A - cI)P = 0$ for all $g \in G$ which implies that $D(g)P$ maps vectors into the null space $\mathcal{N}(A - cI)$. This null space therefore is a subspace that is invariant under $D(g)$, which means that D is reducible unless the null space $\mathcal{N}(A - cI)$ is the whole space. Since by assumption D is irreducible, it follows that $\mathcal{N}(A - cI)$ is the whole space, that is, that $A = cI$. (Issai Schur, 1875–1941)

Example 10.9 (Schur, Wigner, and Eckart) Suppose an arbitrary observable O is invariant under the action of the rotation group $SU(2)$ represented by unitary operators $U(g)$ for $g \in SU(2)$

$$U^\dagger(g)OU(g) = O \quad \text{or} \quad [O, U(g)] = 0. \tag{10.23}$$

These unitary rotation operators commute with the square $\boldsymbol{J}^2$ of the angular momentum $[\boldsymbol{J}^2, U] = 0$. Suppose that they also leave the hamiltonian H unchanged $[H, U] = 0$. Then as shown in example 10.7, the state $U|E, j, m\rangle$ is a sum of states all with the same values of j and E. It follows that

$$\begin{aligned}\sum_{m'} &\langle E, j, m|O|E', j', m'\rangle\langle E', j', m'|U(g)|E', j', m''\rangle \\ &= \sum_{m'}\langle E, j, m|U(g)|E, j, m'\rangle\langle E, j, m'|O|E', j', m''\rangle\end{aligned} \tag{10.24}$$

or in the notation of (10.13)

$$\sum_{m'} \langle E, j, m|O|E', j', m'\rangle D^{(j')}(g)_{m'm''} = \sum_{m'} D^{(j)}(g)_{mm'} \langle E, j, m'|O|E', j', m''\rangle. \tag{10.25}$$

Now Part 1 of Schur's lemma tells us that the matrix $\langle E, j, m|O|E', j', m'\rangle$ must vanish unless the representations are equivalent, which is to say unless $j = j'$. So we have

$$\sum_{m'} \langle E, j, m|O|E', j, m'\rangle D^{(j)}(g)_{m'm''} = \sum_{m'} D^{(j)}(g)_{mm'} \langle E, j, m'|O|E', j, m''\rangle. \tag{10.26}$$

Now Part 2 of Schur's lemma tells us that the matrix $\langle E, j, m|O|E', j, m'\rangle$ must be a multiple of the identity. Thus the symmetry of O under rotations simplifies the matrix element to

$$\langle E, j, m|O|E', j', m'\rangle = \delta_{jj'} \delta_{mm'} O_j(E, E'). \tag{10.27}$$

This result is a special case of the **Wigner-Eckart theorem** (Eugene Wigner 1902–1995, Carl Eckart 1902–1973). □

10.8 Characters

Suppose the $n \times n$ matrices $D_{ij}(g)$ form a representation of a group $G \ni g$. The **character** $\chi_D(g)$ of the matrix $D(g)$ is the trace

$$\chi_D(g) = \mathrm{Tr} D(g) = \sum_{i=1}^{n} D_{ii}(g). \tag{10.28}$$

Traces are cyclic, that is, $\mathrm{Tr} ABC = \mathrm{Tr} BCA = \mathrm{Tr} CAB$. So if two representations D and D' are equivalent, so that $D'(g) = S^{-1} D(g) S$, then they have the same characters because

$$\chi_{D'}(g) = \mathrm{Tr} D'(g) = \mathrm{Tr}\left(S^{-1} D(g) S\right) = \mathrm{Tr}\left(D(g) S S^{-1}\right) = \mathrm{Tr} D(g) = \chi_D(g). \tag{10.29}$$

If two group elements g_1 and g_2 are in the same conjugacy class, that is, if $g_2 = g g_1 g^{-1}$ for all $g \in G$, then they have the same character in a given representation $D(g)$ because

$$\begin{aligned} \chi_D(g_2) &= \mathrm{Tr} D(g_2) = \mathrm{Tr} D(g g_1 g^{-1}) = \mathrm{Tr}\left(D(g) D(g_1) D(g^{-1})\right) \\ &= \mathrm{Tr}\left(D(g_1) D^{-1}(g) D(g)\right) = \mathrm{Tr} D(g_1) = \chi_D(g_1). \end{aligned} \tag{10.30}$$

10.9 Tensor Products

Suppose $D_1(g)$ is a k-dimensional representation of a group G, and $D_2(g)$ is an n-dimensional representation of the same group. Suppose the vectors $|\ell\rangle$ for $\ell = 1 \dots k$ are the basis vectors of the k-dimensional space V_k on which $D_1(g)$ acts, and that the vectors $|m\rangle$ for $m = 1 \dots n$ are the basis vectors of the n-dimensional space V_n on which $D_2(g)$ acts. The $k \times n$ vectors $|\ell, m\rangle$ are basis vectors for the kn-dimensional tensor-product space V_{kn}. The matrices $D_{D_1 \otimes D_2}(g)$ defined as

$$\langle \ell', m' | D_{D_1 \otimes D_2}(g) | \ell, m \rangle = \langle \ell' | D_1(g) | \ell \rangle \langle m' | D_2(g) | m \rangle \tag{10.31}$$

act in this kn-dimensional space V_{kn} and form a representation of the group G; this tensor-product representation usually is reducible. Many tricks help one to decompose reducible tensor-product representations into direct sums of irreducible representations (Georgi, 1999, chap. 10).

Example 10.10 (Adding Angular Momenta) The addition of angular momenta illustrates both the tensor product and its reduction to a direct sum of irreducible representations. Let $D_{j_1}(g)$ and $D_{j_2}(g)$ respectively be the $(2j_1 + 1) \times (2j_1 + 1)$ and the $(2j_2 + 1) \times (2j_2 + 1)$ representations of the rotation group $SU(2)$. The tensor-product representation $D_{D_{j_1} \otimes D_{j_2}}$

$$\langle m_1', m_2' | D_{D_{j_1} \otimes D_{j_2}} | m_1, m_2 \rangle = \langle m_1' | D_{j_1}(g) | m_1 \rangle \langle m_2' | D_{j_2}(g) | m_2 \rangle \tag{10.32}$$

is reducible into a direct sum of all the irreducible representations of $SU(2)$ from $D_{j_1+j_2}(g)$ down to $D_{|j_1-j_2|}(g)$ in integer steps:

$$D_{D_{j_1} \otimes D_{j_2}} = D_{j_1+j_2} \oplus D_{j_1+j_2-1} \oplus \cdots \oplus D_{|j_1-j_2|+1} \oplus D_{|j_1-j_2|} \tag{10.33}$$

each irreducible representation occurring once in the direct sum. □

Example 10.11 (Adding Two Spins) When one adds $j_1 = 1/2$ to $j_2 = 1/2$, one finds that the tensor-product matrix $D_{D_{1/2} \otimes D_{1/2}}$ is equivalent to the direct sum $D_1 \oplus D_0$

$$D_{D_{1/2} \otimes D_{1/2}}(\boldsymbol{\theta}) = S^{-1} \begin{pmatrix} D_1(\boldsymbol{\theta}) & 0 \\ 0 & D_0(\boldsymbol{\theta}) \end{pmatrix} S \tag{10.34}$$

where the matrices S, D_1, and D_0 respectively are 4×4, 3×3, and 1×1. □