9.2 Spherical Bessel functions of the first kind

\[ j_1(x) = \sin x/x^2 - \cos x/x. \]

Rayleigh's formula leads to the recursion relation (exercise 9.22)

\[ j_{\ell+1}(x) = \frac{\ell}{x} j_\ell(x) - j'_\ell(x) \]  

(9.69)

with which one can show (exercise 9.23) that the spherical Bessel functions as defined by Rayleigh's formula do satisfy their differential equation (9.66) with \( x = kr \).

The spherical Bessel functions \( j_\ell(kr) \) satisfy the self-adjoint Sturm-Liouville (6.333) equation (9.66)

\[ -r^2 j''_\ell - 2r j'_\ell + \ell(\ell + 1) j_\ell = k^2 r^2 j_\ell \]

(9.70)

with eigenvalue \( k^2 \) and weight function \( \rho = r^2 \). If \( j_\ell(z_{\ell,n}) = 0 \), then the functions \( j_\ell(kr) = j_\ell(z_{\ell,n}r/a) \) vanish at \( r = a \) and form an orthogonal basis

\[ \int_0^a j_\ell(z_{\ell,n}r/a) j_\ell(z_{\ell,m}r/a) r^2 \, dr = \frac{a^3}{2} j_{\ell+1}(z_{\ell,n}) \delta_{n,m} \]

(9.71)

for a self-adjoint system on the interval \([0, a]\). Moreover, since as \( n \to \infty \) the eigenvalues \( k_{\ell,n}^2 = z_{\ell,n}^2/a^2 \approx [(\ell + 1/2)\pi/a]^2 \to \infty \), the eigenfunctions \( j_\ell(z_{\ell,n}r/a) \) also are complete in the mean (section 6.35).

On an infinite interval, the analogous relation is

\[ \int_0^\infty j_\ell(kr) j_\ell(k'r) r^2 \, dr = \frac{\pi}{2k^2} \delta(k - k'). \]

(9.72)

If we write the spherical Bessel function \( j_0(x) \) as the integral

\[ j_0(z) = \frac{\sin z}{z} = \frac{1}{2} \int_{-1}^{1} e^{izx} \, dx \]

(9.73)

and use Rayleigh's formula (9.68), we may find an integral for \( j_\ell(z) \)

\[ j_\ell(z) = (-1)^\ell z^\ell \left( \frac{1}{z} \frac{d}{dz} \right)^\ell \left( \frac{\sin z}{z} \right) = (-1)^\ell z^\ell \left( \frac{1}{z} \frac{d}{dz} \right)^\ell \left( \frac{1}{2} \int_{-1}^{1} e^{izx} \, dx \right) \]

\[ = \frac{z^\ell}{2} \int_{-1}^{1} \frac{(1 - x^2)^\ell}{2^{\ell+1}} e^{-ixz} \, dx = \frac{(-i)^\ell}{2} \int_{-1}^{1} \frac{(1 - x^2)^\ell}{2^{\ell+1}} \frac{d^\ell}{dx^\ell} e^{ixz} \, dx \]

\[ = \frac{(-i)^\ell}{2} \int_{-1}^{1} e^{ixz} \frac{d^\ell}{dx^\ell} (x^2 - 1)^\ell \, dx = \frac{(-i)^\ell}{2} \int_{-1}^{1} P_\ell(x) e^{ixz} \, dx \]

(9.74)

(exercise 9.24) that contains Rodrigues’s formula (8.8) for the Legendre polynomial \( P_\ell(x) \). With \( z = kr \) and \( x = \cos \theta \), this formula

\[ i^\ell j_\ell(kr) = \frac{1}{2} \int_{-1}^{1} P_\ell(\cos \theta) e^{ikr \cos \theta} d\cos \theta \]

(9.75)