at $\rho = r$ as well as the separable wave equations (9.57). The frequencies of the resonant TE modes then are $\omega_{n,m,\ell} = c\sqrt{\frac{z_{n,m}^2}{r^2} + \frac{\pi^2 \ell^2}{h^2}}$.

The TM modes are $B_z = 0$ and

$$E_z = J_n(z_{n,m} \rho/r) e^{in\phi} \sin(\pi \ell z/h) e^{-i\omega t} \quad (9.62)$$

with resonant frequencies $\omega_{n,m,\ell} = c\sqrt{\frac{z_{n,m}^2}{r^2} + \frac{\pi^2 \ell^2}{h^2}}$.

### 9.2 Spherical Bessel functions of the first kind

If in Bessel’s equation (9.4), one sets $n = \ell + 1/2$ and $j_\ell = \sqrt{\pi/2x} J_{\ell+1/2}$, then one may show (exercise 9.21) that

$$x^2 j_\ell''(x) + 2x j_\ell'(x) + [x^2 - \ell(\ell + 1)] j_\ell(x) = 0 \quad (9.63)$$

which is the equation for the spherical Bessel function $j_\ell$.

We saw in example 6.6 that by setting $V(r, \theta, \phi) = R_{k,\ell}(r) \Theta_{\ell,m}(\theta) \Phi_m(\phi)$
we could separate the variables of Helmholtz’s equation $-\Delta V = k^2 V$ in spherical coordinates

$$\frac{r^2 \Delta V}{V} = \frac{r^2 R_{k,\ell}'}{R_{k,\ell}} + \frac{(\sin \theta \Theta_{\ell,m}')'}{\sin \theta \Theta_{\ell,m}} + \frac{\Phi''}{\sin^2 \theta \Phi} = -k^2 r^2. \quad (9.64)$$

Thus if $\Phi_m(\phi) = e^{im\phi}$ so that $\Phi_m'' = -m^2 \Phi_m$, and if $\Theta_{\ell,m}$ satisfies the associated Legendre equation (8.91)

$$\sin \theta (\sin \theta \Theta_{\ell,m}')' + [\ell(\ell + 1) \sin^2 \theta - m^2] \Theta_{\ell,m} = 0 \quad (9.65)$$

then the product $V(r, \theta, \phi) = R_{k,\ell}(r) \Theta_{\ell,m}(\theta) \Phi_m(\phi)$ will obey (9.64) because in view of (9.63) the radial function $R_{k,\ell}(r) = j_\ell(\rho)$ satisfies

$$(r^2 R_{k,\ell}') + [k^2 r^2 - \ell(\ell + 1)] R_{k,\ell} = 0. \quad (9.66)$$

In terms of the spherical harmonic $Y_{\ell,m}(\theta, \phi) = \Theta_{\ell,m}(\theta) \Phi_m(\phi)$, the solution is $V(r, \theta, \phi) = j_\ell(kr) Y_{\ell,m}(\theta, \phi)$.

Rayleigh’s formula gives the spherical Bessel function

$$j_\ell(x) \equiv \sqrt{\frac{\pi}{2x}} J_{\ell+1/2}(x) \quad (9.67)$$

as the $\ell$th derivative of $\sin x/x$

$$j_\ell(x) = (-1)^\ell x^\ell \left( \frac{1}{x} \frac{d}{dx} \right)^\ell \left( \frac{\sin x}{x} \right) \quad (9.68)$$