When $\alpha = 0$, the Helmholtz equation reduces to the Laplace equation $\triangle V = 0$ of electrostatics which the simpler functions

$$V_{k,n}(\rho, \phi, z) = J_n(k\rho) e^{\pm in\phi} e^{\pm kz} \quad \text{and} \quad V_{k,n}(\rho, \phi, z) = J_n(ik\rho) e^{\pm in\phi} e^{\pm ikz} \tag{9.34}$$

satisfy.

The product $i^{-\nu} J_\nu(ik\rho)$ is real and is known as the **modified Bessel function**

$$I_\nu(k\rho) \equiv i^{-\nu} J_\nu(ik\rho). \tag{9.35}$$

It occurs in various solutions of the **diffusion equation** $D\triangle\phi = \dot{\phi}$. The function $V(\rho, \phi, z) = B(\rho)\Phi(\phi)Z(z)$ satisfies

$$\triangle V = \frac{1}{\rho}\left[(\rho V_{,\rho})_{,\rho} + \frac{1}{\rho} V_{,\phi\phi} + \rho V_{,zz}\right] = \alpha^2 V \tag{9.36}$$

if $B(\rho)$ obeys Bessel's equation

$$\rho \frac{d}{d\rho}\left(\rho \frac{dB}{d\rho}\right) - \left((\alpha^2 - k^2)\rho^2 + n^2\right) B = 0 \tag{9.37}$$

and Φ and Z respectively satisfy

$$-\frac{d^2\Phi}{d\phi^2} = n^2 \Phi(\phi) \quad \text{and} \quad \frac{d^2 Z}{dz^2} = k^2 Z(z) \tag{9.38}$$

or if $B(\rho)$ obeys the Bessel equation

$$\rho \frac{d}{d\rho}\left(\rho \frac{dB}{d\rho}\right) - \left((\alpha^2 + k^2)\rho^2 + n^2\right) B = 0 \tag{9.39}$$

and Φ and Z satisfy

$$-\frac{d^2\Phi}{d\phi^2} = n^2 \Phi(\phi) \quad \text{and} \quad \frac{d^2 Z}{dz^2} = -k^2 Z(z). \tag{9.40}$$

In the first case (9.37 & 9.38), the solution V is

$$V_{k,n}(\rho, \phi, z) = I_n(\sqrt{\alpha^2 - k^2}\,\rho) e^{\pm in\phi} e^{\pm kz} \tag{9.41}$$

while in the second case (9.39 & 9.40), it is

$$V_{k,n}(\rho, \phi, z) = I_n(\sqrt{\alpha^2 + k^2}\,\rho) e^{\pm in\phi} e^{\pm ikz}. \tag{9.42}$$

In both cases, n must be an integer if the solution is to be single valued on the full range of ϕ from 0 to 2π.