These integrals (exercise 9.8) give \( J_n(0) = 0 \) for \( n \neq 0 \), and \( J_0(0) = 1 \).

By differentiating the generating function (9.5) with respect to \( u \) and identifying the coefficients of powers of \( u \), one finds the recursion relation

\[
J_{n-1}(z) + J_{n+1}(z) = \frac{2n}{z} J_n(z).
\]  

(9.8)

Similar reasoning after taking the \( z \) derivative gives (exercise 9.10)

\[
J_{n-1}(z) - J_{n+1}(z) = 2J'_n(z).
\]  

(9.9)

By using the gamma function (section 5.12), one may extend Bessel’s equation (9.4) and its solutions \( J_n(z) \) to non-integral values of \( n \)

\[
J_{\nu}(z) = \left( \frac{z}{2} \right)^\nu \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + \nu + 1)} \left( \frac{z}{2} \right)^{2m}.
\]  

(9.10)

Letting \( z = ax \) in (9.4), we arrive (exercise 9.11) at the self-adjoint form (6.307) of Bessel’s equation

\[
- \frac{d}{dx} \left( x \frac{d}{dx} J_n(ax) \right) + \frac{n^2}{x} J_n(ax) = a^2 x J_n(ax).
\]  

(9.11)

In the notation of equation (6.287), \( p(x) = x \), \( a^2 \) is an eigenvalue, and \( \rho(x) = x \) is a weight function. To have a self-adjoint system (section 6.28) on an interval \([0, b]\), we need the boundary condition (6.247)

\[
0 = \left[ p(J_n v' - J'_n v) \right]_0^b = \left[ x(J_n v' - J'_n v) \right]_0^b
\]  

(9.12)

for all functions \( v(x) \) in the domain \( D \) of the system. Since \( p(x) = x \), \( J_0(0) = 1 \), and \( J_n(0) = 0 \) for integers \( n > 0 \), the terms in this boundary condition vanish at \( x = 0 \) as long as the domain consists of functions \( v(x) \) that are twice differentiable on the interval \([0, b]\). To make these terms vanish at \( x = b \), we require that \( J_n(ab) = 0 \) and that \( v(b) = 0 \). So \( ab \) must be a zero \( z_{n,m} \) of \( J_n(z) \), that is \( J_n(ab) = J_n(z_{n,m}) = 0 \). With \( a = z_{n,m}/b \), Bessel’s equation (9.11) is

\[
- \frac{d}{dx} \left( x \frac{d}{dx} J_n(z_{n,m} x/b) \right) + \frac{n^2}{x} J_n(z_{n,m} x/b) = \frac{z_{n,m}^2}{b^2} x J_n(z_{n,m} x/b).
\]  

(9.13)

For fixed \( n \), the eigenvalue \( a^2 = z_{n,m}^2/b^2 \) is different for each positive integer \( m \). Moreover as \( m \to \infty \), the zeros \( z_{n,m} \) of \( J_n(x) \) rise as \( m\pi \) as one might expect since the leading term of the asymptotic form (9.3) of \( J_n(x) \) is proportional to \( \cos(x - n\pi/2 - \pi/4) \) which has zeros at \( m\pi + (n+1)\pi/2 + \pi/4 \). It follows that the eigenvalues \( a^2 \approx (m\pi)^2/b^2 \) increase without limit as \( m \to \infty \) in accordance with the general result of section 6.34. It follows then from