These integrals (exercise 9.8) give $J_n(0) = 0$ for $n \neq 0$, and $J_0(0) = 1$.

By differentiating the generating function (9.5) with respect to u and identifying the coefficients of powers of u, one finds the recursion relation

$$J_{n-1}(z) + J_{n+1}(z) = \frac{2n}{z} J_n(z). \tag{9.8}$$

Similar reasoning after taking the z derivative gives (exercise 9.10)

$$J_{n-1}(z) - J_{n+1}(z) = 2\, J_n'(z). \tag{9.9}$$

By using the gamma function (section 5.12), one may extend Bessel's equation (9.4) and its solutions $J_n(z)$ to non-integral values of n

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\nu+1)} \left(\frac{z}{2}\right)^{2m}. \tag{9.10}$$

Letting $z = a\,x$ in (9.4), we arrive (exercise 9.11) at the self-adjoint form (6.307) of Bessel's equation

$$-\frac{d}{dx}\left(x\frac{d}{dx}J_n(ax)\right) + \frac{n^2}{x}J_n(ax) = a^2 x J_n(ax). \tag{9.11}$$

In the notation of equation (6.287), $p(x) = x$, a^2 is an eigenvalue, and $\rho(x) = x$ is a weight function. To have a self-adjoint system (section 6.28) on an interval $[0, b]$, we need the boundary condition (6.247)

$$0 = \left[p(J_n v' - J_n' v)\right]_0^b = \left[x(J_n v' - J_n' v)\right]_0^b \tag{9.12}$$

for all functions $v(x)$ in the domain D of the system. Since $p(x) = x$, $J_0(0) = 1$, and $J_n(0) = 0$ for integers $n > 0$, the terms in this boundary condition vanish at $x = 0$ as long as the domain consists of functions $v(x)$ that are twice differentiable on the interval $[0, b]$. To make these terms vanish at $x = b$, we require that $J_n(ab) = 0$ and that $v(b) = 0$. So ab must be a zero $z_{n,m}$ of $J_n(z)$, that is $J_n(ab) = J_n(z_{n,m}) = 0$. With $a = z_{n,m}/b$, Bessel's equation (9.11) is

$$-\frac{d}{dx}\left(x\frac{d}{dx}J_n\left(z_{n,m}x/b\right)\right) + \frac{n^2}{x}J_n\left(z_{n,m}x/b\right) = \frac{z_{n,m}^2}{b^2}\, x\, J_n\left(z_{n,m}x/b\right). \tag{9.13}$$

For fixed n, the eigenvalue $a^2 = z_{n,m}^2/b^2$ is different for each positive integer m. Moreover as $m \to \infty$, the zeros $z_{n,m}$ of $J_n(x)$ rise as $m\pi$ as one might expect since the leading term of the asymptotic form (9.3) of $J_n(x)$ is proportional to $\cos(x - n\pi/2 - \pi/4)$ which has zeros at $m\pi + (n+1)\pi/2 + \pi/4$. It follows that the eigenvalues $a^2 \approx (m\pi)^2/b^2$ increase without limit as $m \to \infty$ in accordance with the general result of section 6.34. It follows then from