Instead, we define \( P_{\ell,m} \) in terms of the \( m \)th derivative \( P_{\ell}^{(m)} \) as
\[
P_{\ell,m}(x) = (1 - x^2)^{m/2} P_{\ell}^{(m)}(x)
\] (8.97)
and compute the derivatives
\[
P_{\ell,m}^{(m+1)} = \left( P_{\ell,m}' + \frac{mx P_{\ell,m}}{1 - x^2} \right) (1 - x^2)^{-m/2}
\] (8.98)
\[
P_{\ell,m}^{(m+2)} = \left[ P_{\ell,m}'' + \frac{2mx P_{\ell,m}'}{1 - x^2} + \frac{mP_{\ell,m}}{1 - x^2} + \frac{m(m+2)x^2 P_{\ell,m}}{(1 - x^2)^2} \right] (1 - x^2)^{-m/2}.
\]

When we put these three expressions in equation (8.94), we get the desired ODE (8.92).

Thus the associated Legendre functions are
\[
P_{\ell,m}(x) = (1 - x^2)^{m/2} P_{\ell}^{(m)}(x) = (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_{\ell}(x)
\] (8.99)
They are simple polynomials in \( x = \cos \theta \) and \( \sqrt{1-x^2} = \sin \theta \)
\[
P_{\ell,m}(\cos \theta) = \sin^m \theta \frac{d^m}{d \cos^m \theta} P_{\ell}(\cos \theta).
\] (8.100)

It follows from Rodrigues’s formula (8.8) for the Legendre polynomial \( P_{\ell}(x) \) that \( P_{\ell,m}(x) \) is given by the similar formula
\[
P_{\ell,m}(x) = \frac{(1 - x^2)^{m/2}}{2^{\ell} \ell!} \frac{d^{\ell+m}}{dx^{\ell+m}} (x^2 - 1)^\ell
\] (8.101)
which tells us that under parity \( P_{\ell}^{m}(x) \) changes by \((-1)^{\ell+m}\)
\[
P_{\ell,m}(-x) = (-1)^{\ell+m} P_{\ell,m}(x).
\] (8.102)

Rodrigues’s formula (8.101) for the associated Legendre function makes sense as long as \( \ell + m \geq 0 \). This last condition is the requirement in quantum mechanics that \( m \) not be less than \(-\ell\). And if \( m \) exceeds \( \ell \), then \( P_{\ell,m}(x) \) is given by more than \( 2\ell \) derivatives of a polynomial of degree \( 2\ell \); so \( P_{\ell,m}(x) = 0 \) if \( m > \ell \). This last condition is the requirement in quantum mechanics that \( m \) not be greater than \( \ell \). So we have
\[
-\ell \leq m \leq \ell.
\] (8.103)

One may show that
\[
P_{\ell,-m}(x) = (-1)^m \frac{(\ell - m)!}{(\ell + m)!} P_{\ell,m}(x).
\] (8.104)