Instead, we define $P_{\ell,m}$ in terms of the mth derivative $P_\ell^{(m)}$ as

$$P_{\ell,m}(x) \equiv (1-x^2)^{m/2}\, P_\ell^{(m)}(x) \tag{8.97}$$

and compute the derivatives

$$P_\ell^{(m+1)} = \left(P'_{\ell,m} + \frac{mxP_{\ell,m}}{1-x^2}\right)(1-x^2)^{-m/2} \tag{8.98}$$

$$P_\ell^{(m+2)} = \left[P''_{\ell,m} + \frac{2mxP'_{\ell,m}}{1-x^2} + \frac{mP_{\ell,m}}{1-x^2} + \frac{m(m+2)x^2P_{\ell,m}}{(1-x^2)^2}\right](1-x^2)^{-m/2}.$$

When we put these three expressions in equation (8.94), we get the desired ODE (8.92).

Thus the associated Legendre functions are

$$P_{\ell,m}(x) = (1-x^2)^{m/2}\, P_\ell^{(m)}(x) = (1-x^2)^{m/2}\,\frac{d^m}{dx^m}P_\ell(x) \tag{8.99}$$

They are simple polynomials in $x = \cos\theta$ and $\sqrt{1-x^2} = \sin\theta$

$$P_{\ell,m}(\cos\theta) = \sin^m\theta\,\frac{d^m}{d\cos^m\theta}\,P_\ell(\cos\theta). \tag{8.100}$$

It follows from Rodrigues's formula (8.8) for the Legendre polynomial $P_\ell(x)$ that $P_{\ell,m}(x)$ is given by the similar formula

$$P_{\ell,m}(x) = \frac{(1-x^2)^{m/2}}{2^\ell \ell!}\,\frac{d^{\ell+m}}{dx^{\ell+m}}\,(x^2-1)^\ell \tag{8.101}$$

which tells us that under parity $P_\ell^m(x)$ changes by $(-1)^{\ell+m}$

$$P_{\ell,m}(-x) = (-1)^{\ell+m}\,P_{\ell,m}(x). \tag{8.102}$$

Rodrigues's formula (8.101) for the associated Legendre function makes sense as long as $\ell + m \geq 0$. This last condition is the requirement in quantum mechanics that m not be less than $-\ell$. And if m exceeds ℓ, then $P_{\ell,m}(x)$ is given by more than 2ℓ derivatives of a polynomial of degree 2ℓ; so $P_{\ell,m}(x) = 0$ if $m > \ell$. This last condition is the requirement in quantum mechanics that m not be greater than ℓ. So we have

$$-\ell \leq m \leq \ell. \tag{8.103}$$

One may show that

$$P_{\ell,-m}(x) = (-1)^m\frac{(\ell-m)!}{(\ell+m)!}\,P_{\ell,m}(x). \tag{8.104}$$