The generating function \( g(t, x) \) is even under the reflection of both independent variables, so

\[
g(t, x) = \sum_{n=0}^{\infty} t^n P_n(x) = \sum_{n=0}^{\infty} (-t)^n P_n(-x) = g(-t, -x)
\]

which implies that

\[
P_n(-x) = (-1)^n P_n(x) \quad \text{whence} \quad P_{2n+1}(0) = 0.
\]

With more effort, one can show that

\[
P_{2n}(0) = (-1)^n \frac{(2n-1)!!}{(2n)!!} \quad \text{and that} \quad |P_n(x)| \leq 1.
\]

8.7 Schlaefli’s Integral

Schlaefli used Cauchy’s integral formula (5.36) and Rodrigues’s formula

\[
P_n(x) = \frac{1}{2^n n!} \left( \frac{d}{dx} \right)^n (x^2 - 1)^n
\]

to express \( P_n(z) \) as a counterclockwise contour integral around the point \( z \)

\[
P_n(z) = \frac{1}{2^n 2\pi i} \oint \frac{(z'^2 - 1)^n}{(z' - z)^{n+1}} dz'.
\]

8.8 Orthogonal Polynomials

Rodrigues’s formula (8.8) generates other families of orthogonal polynomials. The \( n \)-th order polynomials \( R_n \) in which the \( e_n \) are constants

\[
R_n(x) = \frac{1}{e_n w(x)} \frac{d^n}{dx^n} [w(x) Q^n(x)]
\]

are orthogonal on the interval from \( a \) to \( b \) with weight function \( w(x) \)

\[
\int_a^b R_n(x) R_k(x) w(x) \, dx = N_n \delta_{nk}
\]

as long as \( Q(x) \) vanishes at \( a \) and \( b \) (exercise 8.8)

\[
Q(a) = Q(b) = 0.
\]