

Exercises

6.1 In rectangular coordinates, the curl of a curl is by definition (6.40)

$$(\nabla \times (\nabla \times \mathbf{E}))_i = \sum_{j,k=1}^3 \epsilon_{ijk} \partial_j (\nabla \times \mathbf{E})_k = \sum_{j,k,\ell,m=1}^3 \epsilon_{ijk} \partial_j \epsilon_{k\ell m} \partial_\ell E_m. \quad (6.436)$$

Use Levi-Civita's identity (1.449) to show that

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \Delta \mathbf{E}. \quad (6.437)$$

This formula defines $\Delta \mathbf{E}$ in any system of orthogonal coordinates.

6.2 Show that since the Bessel function $J_n(x)$ satisfies Bessel's equation (6.48), the function $P_n(\rho) = J_n(k\rho)$ satisfies (6.47).

6.3 Show that (6.58) implies that $R_{k,\ell}(r) = j_\ell(kr)$ satisfies (6.57).

6.4 Use (6.56, 6.57), and $\Phi_m'' = -m^2 \Phi_m$ to show in detail that the product $f(r, \theta, \phi) = R_{k,\ell}(r) \Theta_{\ell,m}(\theta) \Phi_m(\phi)$ satisfies $-\Delta f = k^2 f$.

6.5 Replacing Helmholtz's k^2 by $2m(E - V(r))/\hbar^2$, we get Schrödinger's equation

$$-(\hbar^2/2m)\Delta\psi(r, \theta, \phi) + V(r)\psi(r, \theta, \phi) = E\psi(r, \theta, \phi). \quad (6.438)$$

Let $\psi(r, \theta, \phi) = R_{n,\ell}(r)\Theta_{\ell,m}(\theta)e^{im\phi}$ in which $\Theta_{\ell,m}$ satisfies (6.56) and show that the radial function $R_{n,\ell}$ must obey

$$-(r^2 R_{n,\ell}')'/r^2 + [\ell(\ell+1)/r^2 + 2mV/\hbar^2] R_{n,\ell} = 2mE_{n,\ell} R_{n,\ell}/\hbar^2. \quad (6.439)$$

6.6 Use the empty-space Maxwell's equations $\nabla \cdot \mathbf{B} = 0$, $\nabla \times \mathbf{E} + \dot{\mathbf{B}} = 0$, $\nabla \cdot \mathbf{E} = 0$, and $\nabla \times \mathbf{B} - \dot{\mathbf{E}}/c^2 = 0$ and the formula (6.437) to show that in vacuum $\Delta \mathbf{E} = \ddot{\mathbf{E}}/c^2$ and $\Delta \mathbf{B} = \ddot{\mathbf{B}}/c^2$.

6.7 Argue from symmetry and anti-symmetry that $[\gamma^a, \gamma^b] \partial_a \partial_b = 0$ in which the sums over a and b run from 0 to 3.

6.8 Suppose a voltage $V(t) = V \sin(\omega t)$ is applied to a resistor of R (Ω) in series with a capacitor of capacitance C (F). If the current through the circuit at time $t = 0$ is zero, what is the current at time t ?

6.9 (a) Is $(1 + x^2 + y^2)^{-3/2} [(1 + y^2)y dx + (1 + x^2)x dy] = 0$ exact? (b) Find its general integral and solution $y(x)$. Use section 6.11.

6.10 (a) Separate the variables of the ODE $(1 + y^2)y dx + (1 + x^2)x dy = 0$. (b) Find its general integral and solution $y(x)$.

6.11 Find the general solution to the differential equation $y' + y/x = c/x$.

6.12 Find the general solution to the differential equation $y' + xy = ce^{-x^2/2}$.

- 6.13 James Bernoulli studied ODEs of the form $y' + p y = q y^n$ in which p and q are functions of x . Division by y^n and the substitution $v = y^{1-n}$ gives us the equation $v' + (1-n)p v = (1-n)q$ which is soluble as shown in section (6.16). Use this method to solve the ODE $y' - y/2x = 5x^2y^5$.
- 6.14 Integrate the ODE $(xy + 1) dx + 2x^2(2xy - 1) dy = 0$. Hint: Use the variable $v(x) = xy(x)$ instead of $y(x)$.
- 6.15 Show that the points $x = \pm 1$ and ∞ are regular singular points of Legendre's equation (6.181).
- 6.16 Use the vanishing of the coefficient of every power of x in (6.185) and the notation (6.187) to derive the recurrence relation (6.188).
- 6.17 In example 6.29, derive the recursion relation for $r = 1$ and discuss the resulting eigenvalue equation.
- 6.18 In example 6.29, show that the solutions associated with the roots $r = 0$ and $r = 1$ are the same.
- 6.19 For a hydrogen atom, we set $V(r) = -e^2/4\pi\epsilon_0 r \equiv -q^2/r$ in (6.439) and get $(r^2 R'_{n,\ell})' + [(2m/\hbar^2)(E_{n,\ell} + Zq^2/r)r^2 - \ell(\ell + 1)] R_{n,\ell} = 0$. So at big r , $R''_{n,\ell} \approx -2mE_{n,\ell}R_{n,\ell}/\hbar^2$ and $R_{n,\ell} \sim \exp(-\sqrt{-2mE_{n,\ell}}r/\hbar)$. At tiny r , $(r^2 R'_{n,\ell})' \approx \ell(\ell + 1)R_{n,\ell}$ and $R_{n,\ell}(r) \sim r^\ell$. Set $R_{n,\ell}(r) = r^\ell \exp(-\sqrt{-2mE_{n,\ell}}r/\hbar)P_{n,\ell}(r)$ and apply the method of Frobenius to find the values of $E_{n,\ell}$ for which $R_{n,\ell}$ is suitably normalizable.
- 6.20 Show that as long as the matrix $\mathcal{Y}_{kj} = y_k^{(\ell_j)}(x_j)$ is nonsingular, the n boundary conditions

$$b_j = y^{(\ell_j)}(x_j) = \sum_{k=1}^n c_k y_k^{(\ell_j)}(x_j) \quad (6.440)$$

determine the n coefficients c_k of the expansion (6.222) to be

$$C^T = B \mathcal{Y}^{-1} \quad \text{or} \quad C_k = \sum_{j=1}^n b_j \mathcal{Y}_{jk}^{-1}. \quad (6.441)$$

- 6.21 Show that if the real and imaginary parts u_1, u_2, v_1 , and v_2 of ψ and χ satisfy boundary conditions at $x = a$ and $x = b$ that make the boundary term (6.235) vanish, then its complex analog (6.242) also vanishes.
- 6.22 Show that if the real and imaginary parts u_1, u_2, v_1 , and v_2 of ψ and χ satisfy boundary conditions at $x = a$ and $x = b$ that make the boundary term (6.235) vanish, and if the differential operator L is real and self adjoint, then (6.238) implies (6.243).
- 6.23 Show that if D is the set of all twice-differentiable functions $u(x)$ on $[a, b]$ that satisfy Dirichlet's boundary conditions (6.245) and if the function $p(x)$ is continuous and positive on $[a, b]$, then the adjoint set