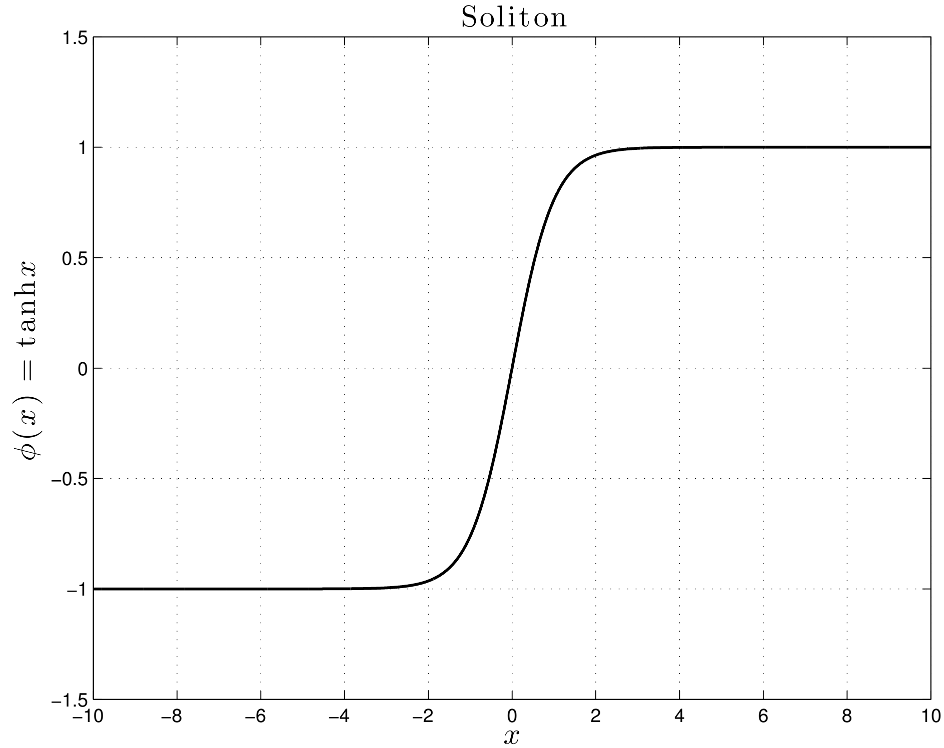


Figure 6.4 The field $\phi(x)$ of the soliton (6.435) at rest ($v = 0$) at position $x_0 = 0$ for $\lambda = 1 = \phi_0$. The energy density of the field vanishes when $\phi = \pm\phi_0 = \pm 1$. The energy of this soliton is concentrated at $x = 0$.

in which $C$ is a constant of integration. □

The equations of particle physics are nonlinear. Physicists usually use perturbation theory to cope with the nonlinearities. But occasionally they focus on the nonlinearities and treat the fields classically or semi-classically. To keep things relatively simple, we'll work in a space-time of only two dimensions and consider a model field theory described by the action density

$$\mathcal{L} = \frac{1}{2}\left(\dot{\phi}^2 - \phi'^2\right) - V(\phi) \tag{6.427}$$

in which $V$ is a simple function of the field $\phi$. Lagrange's equation for this theory is

$$\ddot{\phi} - \phi'' = -\frac{dV}{d\phi}. \tag{6.428}$$

We can convert this partial differential equation to an ordinary one by making the field $\phi$ depend only upon the combination $u = x - v\,t$ rather than upon both $x$ and $t$. We then have $\dot{\phi} = \; - v\,\phi_u$. With this restriction to traveling-wave solutions, Lagrange's equation reduces to

$$(1 - v^2)\,\phi_{uu} = \frac{dV}{d\phi}. \tag{6.429}$$

We multiply both sides of this equation by $\phi_u$

$$(1 - v^2)\,\phi_u\,\phi_{uu} = \frac{dV}{d\phi}\,\phi_u \tag{6.430}$$

and integrate both sides to get $(1 - v^2)\,\frac{1}{2}\,\phi_u^2 = V + E$ in which $E$ is a constant of integration

$$E = \tfrac{1}{2}\,(1 - v^2)\;\phi_u^2 - V(\phi). \tag{6.431}$$

We can convert (exercise 6.37) this equation into a problem of integration

$$u - u_0 = \int \frac{\sqrt{1 - v^2}}{\sqrt{2(E + V(\phi))}}\;d\phi. \tag{6.432}$$

By inverting the resulting equation relating $u$ to $\phi$, we may find the **soliton** solution $\phi(u - u_0)$, which is a lump of energy traveling with speed $v$.

**Example 6.48** (Soliton of the $\phi^4$ Theory) To simplify the integration (6.432), we take as the action density

$$\mathcal{L} = \frac{1}{2}\left(\dot{\phi}^2 - \phi'^2\right) - \left[\frac{\lambda^2}{2}\left(\phi^2 - \phi_0^2\right)^2 \; - E\right]. \tag{6.433}$$

Our formal solution (6.432) gives

$$u - u_0 = \pm \int \frac{\sqrt{1 - v^2}}{\lambda\left(\phi^2 - \phi_0^2\right)}\;d\phi = \mp \frac{\sqrt{1 - v^2}}{\lambda \phi_0}\;\tanh^{-1}(\phi/\phi_0) \tag{6.434}$$

or

$$\phi(x - vt) = \mp\phi_0\;\tanh\left[\lambda\phi_0 \frac{x - x_0 - v(t - t_0)}{\sqrt{1 - v^2}}\right] \tag{6.435}$$

which is a soliton (or an antisoliton) at $x_0 + v(t - t_0)$. A unit soliton at rest is plotted in Fig. 6.4. Its energy is concentrated at $x = 0$ where $|\phi^2 - \phi_0^2|$ is maximal. □

**Exercises**

6.1 In rectangular coordinates, the curl of a curl is by definition (6.40)

$$(\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{E}))_i = \sum_{j,k=1}^{3} \epsilon_{ijk} \partial_j (\boldsymbol{\nabla} \times \boldsymbol{E})_k = \sum_{j,k,\ell,m=1}^{3} \epsilon_{ijk} \partial_j \epsilon_{k\ell m} \partial_\ell E_m. \tag{6.436}$$

Use Levi-Civita's identity (1.449) to show that

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{E}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{E}) - \triangle \boldsymbol{E}. \tag{6.437}$$

This formula defines $\triangle \boldsymbol{E}$ in any system of orthogonal coordinates.

6.2 Show that since the Bessel function $J_n(x)$ satisfies Bessel's equation (6.48), the function $P_n(\rho) = J_n(k\rho)$ satisfies (6.47).

6.3 Show that (6.58) implies that $R_{k,\ell}(r) = j_\ell(kr)$ satisfies (6.57).

6.4 Use (6.56, 6.57), and $\Phi_m'' = -m^2 \Phi_m$ to show in detail that the product $f(r,\theta,\phi) = R_{k,\ell}(r)\,\Theta_{\ell,m}(\theta)\,\Phi_m(\phi)$ satisfies $-\triangle f = k^2 f$.

6.5 Replacing Helmholtz's $k^2$ by $2m(E - V(r))/\hbar^2$, we get Schrödinger's equation

$$-(\hbar^2/2m)\triangle\psi(r,\theta,\phi) + V(r)\psi(r,\theta,\phi) = E\psi(r,\theta,\phi). \tag{6.438}$$

Let $\psi(r,\theta,\phi) = R_{n,\ell}(r)\Theta_{\ell,m}(\theta)e^{im\phi}$ in which $\Theta_{\ell,m}$ satisfies (6.56) and show that the radial function $R_{n,\ell}$ must obey

$$-\left(r^2 R_{n,\ell}'\right)'/r^2 + \left[\ell(\ell+1)/r^2 + 2mV/\hbar^2\right] R_{n,\ell} = 2mE_{n,\ell}\, R_{n,\ell}/\hbar^2. \tag{6.439}$$

6.6 Use the empty-space Maxwell's equations $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$, $\boldsymbol{\nabla} \times \boldsymbol{E} + \dot{\boldsymbol{B}} = 0$, $\boldsymbol{\nabla} \cdot \boldsymbol{E} = 0$, and $\boldsymbol{\nabla} \times \boldsymbol{B} - \dot{\boldsymbol{E}}/\boldsymbol{c^2} = 0$ and the formula (6.437) to show that in vacuum $\triangle \boldsymbol{E} = \ddot{\boldsymbol{E}}/c^2$ and $\triangle \boldsymbol{B} = \ddot{\boldsymbol{B}}/c^2$.

6.7 Argue from symmetry and anti-symmetry that $[\gamma^a, \gamma^b]\partial_a\partial_b = 0$ in which the sums over $a$ and $b$ run from 0 to 3.

6.8 Suppose a voltage $V(t) = V\sin(\omega t)$ is applied to a resistor of $R$ $(\Omega)$ in series with a capacitor of capacitance $C$ $(F)$. If the current through the circuit at time $t = 0$ is zero, what is the current at time $t$?

6.9 (a) Is $\left(1 + x^2 + y^2\right)^{-3/2}\left[(1+y^2)y\,dx + (1+x^2)x\,dy\right] = 0$ exact? (b) Find its general integral and solution $y(x)$. Use section 6.11.

6.10 (a) Separate the variables of the ODE $(1+y^2)y\,dx + (1+x^2)x\,dy = 0$. (b) Find its general integral and solution $y(x)$.

6.11 Find the general solution to the differential equation $y' + y/x = c/x$.

6.12 Find the general solution to the differential equation $y' + xy = ce^{-x^2/2}$.

6.13 James Bernoulli studied ODEs of the form $y' + p\,y = q\,y^n$ in which $p$ and $q$ are functions of $x$. Division by $y^n$ and the substitution $v = y^{1-n}$ gives us the equation $v' + (1-n)p\,v = (1-n)\,q$ which is soluble as shown in section (6.16). Use this method to solve the ODE $y' - y/2x = 5x^2y^5$.

6.14 Integrate the ODE $(xy+1)\,dx + 2x^2(2xy-1)\,dy = 0$. Hint: Use the variable $v(x) = xy(x)$ instead of $y(x)$.

6.15 Show that the points $x = \pm 1$ and $\infty$ are regular singular points of Legendre's equation (6.181).

6.16 Use the vanishing of the coefficient of every power of $x$ in (6.185) and the notation (6.187) to derive the recurrence relation (6.188).

6.17 In example 6.29, derive the recursion relation for $r = 1$ and discuss the resulting eigenvalue equation.

6.18 In example 6.29, show that the solutions associated with the roots $r = 0$ and $r = 1$ are the same.

6.19 For a hydrogen atom, we set $V(r) = -e^2/4\pi\epsilon_0 r \equiv -q^2/r$ in (6.439) and get $(r^2 R'_{n,\ell})' + \left[(2m/\hbar^2)\left(E_{n,\ell} + Zq^2/r\right) r^2 - \ell(\ell+1)\right] R_{n,\ell} = 0$. So at big $r$, $R''_{n,\ell} \approx -2mE_{n,\ell}R_{n,\ell}/\hbar^2$ and $R_{n,\ell} \sim \exp(-\sqrt{-2mE_{n,\ell}}\,r/\hbar)$. At tiny $r$, $(r^2R'_{n,\ell})' \approx \ell(\ell+1)R_{n,\ell}$ and $R_{n,\ell}(r) \sim r^\ell$. Set $R_{n,\ell}(r) = r^\ell \exp(-\sqrt{-2mE_{n,\ell}}\,r/\hbar)P_{n,\ell}(r)$ and apply the method of Frobenius to find the values of $E_{n,\ell}$ for which $R_{n,\ell}$ is suitably normalizable.

6.20 Show that as long as the matrix $\mathcal{Y}_{kj} = y_k^{(\ell_j)}(x_j)$ is nonsingular, the $n$ boundary conditions

$$b_j = y^{(\ell_j)}(x_j) = \sum_{k=1}^{n} c_k\, y_k^{(\ell_j)}(x_j) \tag{6.440}$$

determine the $n$ coefficients $c_k$ of the expansion (6.222) to be

$$C^{\mathsf{T}} = B^{\mathsf{T}}\,\mathcal{Y}^{-1} \quad \text{or} \quad C_k = \sum_{j=1}^{n} b_j \mathcal{Y}^{-1}_{jk}. \tag{6.441}$$

6.21 Show that if the real and imaginary parts $u_1$, $u_2$, $v_1$, and $v_2$ of $\psi$ and $\chi$ satisfy boundary conditions at $x = a$ and $x = b$ that make the boundary term (6.235) vanish, then its complex analog (6.242) also vanishes.

6.22 Show that if the real and imaginary parts $u_1$, $u_2$, $v_1$, and $v_2$ of $\psi$ and $\chi$ satisfy boundary conditions at $x = a$ and $x = b$ that make the boundary term (6.235) vanish, and if the differential operator $L$ is real and self adjoint, then (6.238) implies (6.243).

6.23 Show that if $D$ is the set of all twice-differentiable functions $u(x)$ on $[a, b]$ that satisfy Dirichlet's boundary conditions (6.245) and if the function $p(x)$ is continuous and positive on $[a, b]$, then the adjoint set