(Carl Neumann, 1832–1925).

6.27 Self-Adjoint Differential Operators

If $p(x)$ and $q(x)$ are real, then the differential operator

$$L = -\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x) \tag{6.233}$$

is formally **self adjoint**. Such operators are interesting because if we take any two functions u and v that are twice differentiable on an interval $[a, b]$ and integrate $v\,L\,u$ twice by parts over the interval, we get

$$\begin{aligned}(v, L\,u) &= \int_a^b v\,L\,u\,dx = \int_a^b v\left[-\left(pu'\right)' + qu\right]\,dx \\ &= \int_a^b \left[pu'v' + uqv\right]\,dx - \left[vpu'\right]_a^b \\ &= \int_a^b \left[-(pv')' + qv\right]\,u\,dx + \left[puv' - vpu'\right]_a^b \\ &= \int_a^b (L\,v)\,u\,dx + \left[p(uv' - vu')\right]_a^b\end{aligned} \tag{6.234}$$

which is **Green's formula**

$$\int_a^b (vL\,u - u\,L\,v)\,dx = \left[p(uv' - vu')\right]_a^b = [pW(u,v)]_a^b \tag{6.235}$$

(George Green, 1793–1841). Its differential form is **Lagrange's identity**

$$vL\,u - u\,L\,v = \left[p\,W(u,v)\right]' \tag{6.236}$$

(Joseph-Louis Lagrange, 1736–1813). Thus if the twice-differentiable functions u and v satisfy boundary conditions at $x = a$ and $x = b$ that make the boundary term (6.235) vanish

$$\left[p(uv' - vu')\right]_a^b = [pW(u,v)]_a^b = 0 \tag{6.237}$$

then the real differential operator L is **symmetric**

$$(v, L\,u) = \int_a^b v\,L\,u\,dx = \int_a^b u\,L\,v\,dx = (u, L\,v). \tag{6.238}$$

A real linear operator A that acts in a real vector space and satisfies the analogous relation (1.161)

$$(g, A\,f) = (f, A\,g) \tag{6.239}$$