in which $a_0 \neq 0$ is the coefficient of the lowest power of x in $y(x)$. Differentiating, we have

$$y'(x) = \sum_{n=0}^{\infty} (r+n)\, a_n\, x^{r+n-1} \tag{6.183}$$

and

$$y''(x) = \sum_{n=0}^{\infty} (r+n)(r+n-1)\, a_n\, x^{r+n-2}. \tag{6.184}$$

When we substitute the three series (6.182–6.184) into our differential equation $x^2 y'' + xp(x)y' + q(x)y = 0$, we find

$$\sum_{n=0}^{\infty} \left[(n+r)(n+r-1) + (n+r)p(x) + q(x)\right] a_n x^{n+r} = 0. \tag{6.185}$$

If this equation is to be satisfied for all x, then the coefficient of every power of x must vanish. The lowest power of x is x^r, and it occurs when $n = 0$ with coefficient $[r(r-1+p(0)) + q(0)]\, a_0$. Thus since $a_0 \neq 0$, we have

$$r(r-1+p(0)) + q(0) = 0. \tag{6.186}$$

This quadratic **indicial equation** has two roots r_1 and r_2.

To analyze higher powers of x, we introduce the notation

$$p(x) = \sum_{j=0}^{\infty} p_j x^j \quad \text{and} \quad q(x) = \sum_{j=0}^{\infty} q_j x^j \tag{6.187}$$

in which $p_0 = p(0)$ and $q_0 = q(0)$. The requirement (exercise 6.16) that the coefficient of x^{r+k} vanish gives us a **recurrence relation**

$$a_k = -\left[\frac{1}{(r+k)(r+k-1+p_0) + q_0}\right] \sum_{j=0}^{k-1} \left[(j+r)p_{k-j} + q_{k-j}\right] a_j \tag{6.188}$$

that expresses a_k in terms of $a_0,\ a_1,\ \ldots a_{k-1}$. When $p(x)$ and $q(x)$ are polynomials of low degree, these equations become much simpler.

Example 6.28 (Sines and Cosines) To apply Frobenius's method the ODE $y'' + \omega^2 y = 0$, we first write it in the form $x^2 y'' + xp(x)y' + q(x)y = 0$ in which $p(x) = 0$ and $q(x) = \omega^2 x^2$. So both $p(0) = p_0 = 0$ and $q(0) = q_0 = 0$, and the indicial equation (6.186) is $r(r-1) = 0$ with roots $r_1 = 0$ and $r_2 = 1$.

We first set $r = r_1 = 0$. Since the p's and q's vanish except for $q_2 = \omega^2$, the recurrence relation (6.188) is $a_k = -q_2\, a_{k-2}/k(k-1) = -\omega^2 a_{k-2}/k(k-1)$. Thus $a_2 = -\omega^2 a_0/2$, and $a_{2n} = (-1)^n \omega^{2n} a_0/(2n)!$. The recurrence relation (6.188) gives no information about a_1, so to find the simplest solution, we set $a_1 = 0$. The recurrence relation $a_k = -\omega^2 a_{k-2}/k(k-1)$ then makes all the terms a_{2n+1} of odd index vanish. Our solution for the first root $r_1 = 0$ then is

$$y(x) = \sum_{n=0}^{\infty} a_n\, x^n = a_0 \sum_{n=0}^{\infty} (-1)^n \frac{(\omega x)^{2n}}{(2n)!} = a_0\, \cos \omega x. \tag{6.189}$$

Similarly, the recurrence relation (6.188) for the second root $r_2 = 1$ is $a_k = -\omega^2 a_{k-2}/k(k+1)$, so that $a_{2n} = (-1)^n \omega^{2n} a_0/(2n+1)!$, and we again set all the terms of odd index equal to zero. Thus we have

$$y(x) = x \sum_{n=0}^{\infty} a_n\, x^n = \frac{a_0}{\omega} \sum_{n=0}^{\infty} (-1)^n \frac{(\omega x)^{2n+1}}{(2n+1)!} = \frac{a_0}{\omega}\, \sin \omega x \tag{6.190}$$

as our solution for the second root $r_2 = 1$. □

Frobenius's method sometimes shows that solutions exist only when a parameter in the ODE assumes a special value called an **eigenvalue**.

Example 6.29 (Legendre's Equation) If one rewrites Legendre's equation $(1-x^2)y'' - 2xy' + \lambda y = 0$ as $x^2 y'' + xpy' + qy = 0$, then one finds $p(x) = -2x^2/(1-x^2)$ and $q(x) = x^2\lambda/(1-x^2)$, which are analytic but not polynomials. In this case, it is simpler to substitute the expansions (6.182–6.184) directly into Legendre's equation $(1-x^2)y'' - 2xy' + \lambda y = 0$. We then find

$$\sum_{n=0}^{\infty} \left[(n+r)(n+r-1)(1-x^2)x^{n+r-2} - 2(n+r)x^{n+r} + \lambda x^{n+r}\right] a_n = 0.$$

The coefficient of the lowest power of x is $r(r-1)a_0$, and so the indicial equation is $r(r-1) = 0$. For $r = 0$, we shift the index n on the term $n(n-1)x^{n-2}a_n$ to $n = j+2$ and replace n by j in the other terms:

$$\sum_{j=0}^{\infty} \left\{(j+2)(j+1)\, a_{j+2} - [j(j-1) + 2j - \lambda]\, a_j\right\} x^j = 0. \tag{6.191}$$

Since the coefficient of x^j must vanish, we get the recursion relation

$$a_{j+2} = \frac{j(j+1) - \lambda}{(j+2)(j+1)} a_j \tag{6.192}$$

which for big j says that $a_{j+2} \approx a_j$. Thus the series (6.182) does not converge for $|x| \geq 1$ unless $\lambda = j(j+1)$ for some integer j in which case the series (6.182) is a Legendre polynomial (chapter 8). □