5.12 Evaluate the contour integral of the function $f(z) = \sin wz/(z-5)^3$ along the curve $z = 6 + 4(\cos t + i \sin t)$ for $0 \le t \le 2\pi$.

5.13 Evaluate the contour integral of the function $f(z) = \sin wz/(z-5)^3$ along the curve $z = -6 + 4(\cos t + i \sin t)$ for $0 \le t \le 2\pi$.

5.14 Is the function $f(x,y) = x^2 + iy^2$ analytic?

5.15 Is the function $f(x,y) = x^3 - 3xy^2 + 3ix^2y - iy^3$ analytic? Is the function $x^3 - 3xy^2$ harmonic? Does it have a minimum or a maximum? If so, what are they?

5.16 Is the function $f(x,y) = x^2 + y^2 + i(x^2 + y^2)$ analytic? Is $x^2 + y^2$ a harmonic function? What is its minimum, if it has one?

5.17 Derive the first three nonzero terms of the Laurent series for $f(z) = 1/(e^z - 1)$ about $z = 0$.

5.18 Assume that a function $g(z)$ is meromorphic in R and has a Laurent series (5.97) about a point $w \in R$. Show that as $z \to w$, the ratio $g'(z)/g(z)$ becomes (5.95).

5.19 Find the poles and residues of the functions $1/\sin z$ and $1/\cos z$.

5.20 Derive the integral formula (5.122) from (5.121).

5.21 Show that if $\mathrm{Re}\, w < 0$, then for arbitrary complex z

$$\int_{-\infty}^{\infty} e^{w(x+z)^2}\, dx = \sqrt{\frac{\pi}{-w}}. \tag{5.347}$$

5.22 Use a ghost contour to evaluate the integral

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2}\, dx.$$

Show your work; do not just quote the result of a commercial math program.

5.23 For $a > 0$ and $b^2 - 4ac < 0$, use a ghost contour to do the integral

$$\int_{-\infty}^{\infty} \frac{dx}{ax^2 + bx + c}. \tag{5.348}$$

5.24 Show that

$$\int_0^{\infty} \cos ax\, e^{-x^2}\, dx = \frac{1}{2}\sqrt{\pi}\, e^{-a^2/4}. \tag{5.349}$$

5.25 Show that

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^4} = \frac{\pi}{\sqrt{2}}. \tag{5.350}$$

5.26 Evaluate the integral

$$\int_0^{\infty} \frac{\cos x}{1 + x^4}\, dx. \tag{5.351}$$

5.27 Show that the Yukawa Green's function (5.151) reproduces the Yukawa potential (5.141) when $n = 3$. Use $K_{1/2}(x) = \sqrt{\pi/2x}\, e^{-x}$ (9.99).

5.28 Derive the two explicit formulas (5.188) and (5.189) for the square-root of a complex number.

5.29 What is $(-i)^i$? What is the most general value of this expression?

5.30 Use the indefinite integral (5.223) to derive the principal-part formula (5.224).

5.31 The Bessel function $J_n(x)$ is given by the integral

$$J_n(x) = \frac{1}{2\pi i} \oint_C e^{(x/2)(z - 1/z)} \frac{dz}{z^{n+1}} \tag{5.352}$$

along a counter-clockwise contour about the origin. Find the generating function for these Bessel functions, that is, the function $G(x, z)$ whose Laurent series has the $J_n(x)$'s as coefficients

$$G(x, z) = \sum_{n=-\infty}^{\infty} J_n(x)\, z^n. \tag{5.353}$$

5.32 Show that the Heaviside function $\theta(y) = (y + |y|)/(2|y|)$ is given by the integral

$$\theta(y) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{iyx} \frac{dx}{x - i\epsilon} \tag{5.354}$$

in which ϵ is an infinitesimal positive number.

5.33 Show that the integral of $\exp(ik)/k$ along the contour from $k = L$ to $k = L + iH$ and then to $k = -L + iH$ and then down to $k = -L$ vanishes in the double limit $L \to \infty$ and $H \to \infty$.

5.34 Use a ghost contour and a cut to evaluate the integral

$$I = \int_{-1}^{1} \frac{dx}{(x^2 + 1)\sqrt{1 - x^2}} \tag{5.355}$$

by imitating example 5.30. Be careful when picking up the poles at $z = \pm i$. If necessary, use the explicit square-root formulas (5.188) and (5.189).

5.35 Redo the previous exercise (5.34) by defining the square roots so that the cuts run from $-\infty$ to -1 and from 1 to ∞. Take advantage of the evenness of the integrand and integrate on a contour that is slightly above the whole real axis. Then add a ghost contour around the upper half plane.

5.36 Show that if u is even and v is odd, then the Hilbert transforms (5.265) imply (5.267).