$T(z)$ are defined by its Laurent series

$$T(z) = \sum_{n=-\infty}^{\infty} \frac{L_n}{z^{n+2}}$$

(5.336)

and the inverse relation

$$L_n = \frac{1}{2\pi i} \oint z^{n+1} T(z) \, dz.$$  

(5.337)

Thus the commutator of two modes involves two loop integrals

$$[L_m, L_n] = \left[ \frac{1}{2\pi i} \oint z^{m+1} T(z) \, dz, \frac{1}{2\pi i} \oint w^{n+1} T(w) \, dw \right]$$

(5.338)

which we may deform as long as we cross no poles. Let’s hold $w$ fixed and deform the $z$ loop so as to keep the $T$’s radially ordered when $z$ is near $w$ as in Fig. 5.10. The operator-product expansion of the radially ordered product $\mathcal{R}\{T(z)T(w)\}$ is

$$\mathcal{R}\{T(z)T(w)\} = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2} T(w) + \frac{1}{z-w} T'(w) + \ldots$$

(5.339)

in which the prime means derivative, $c$ is a constant, and the dots denote terms that are analytic in $z$ and $w$. The commutator introduces a minus sign that cancels most of the two contour integrals and converts what remains into an integral along a tiny circle $C_w$ about the point $w$ as in Fig. 5.10

$$[L_m, L_n] = \oint_{C_w} \frac{dw}{2\pi i} w^{n+1} \oint_{C_w} \frac{dz}{2\pi i} z^{m+1} \left[ \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{T'(w)}{z-w} \right].$$

(5.340)

After doing the $z$-integral, which is left as a homework exercise (5.43), one may use the Laurent series (5.336) for $T(w)$ to do the $w$-integral, which one may choose to be along a tiny circle about $w = 0$, and so find the commutator

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12} m(m^2 - 1) \delta_{m+n,0}$$

(5.341)

of the Virasoro algebra.

Exercises

5.1 Compute the two limits (5.6) and (5.7) of example 5.2 but for the function $f(x, y) = x^2 - y^2 + 2ixy$. Do the limits now agree? Explain.

5.2 Show that if $f(z)$ is analytic in a disk, then the integral of $f(z)$ around a tiny (isosceles) triangle of side $\epsilon \ll 1$ inside the disk is zero to order $\epsilon^2$. 