5.20 Kramers-Kronig Relations

If we use $\sigma \boldsymbol{E}$ for the current density $\boldsymbol{J}$ and $\boldsymbol{E}(t) = e^{-i\omega t}\boldsymbol{E}$ for the electric field, then Maxwell's equation $\boldsymbol{\nabla} \times \boldsymbol{B} = \mu \boldsymbol{J} + \epsilon\mu \dot{\boldsymbol{E}}$ becomes

$$\boldsymbol{\nabla} \times \boldsymbol{B} = -i\omega\epsilon\mu \left(1 + i\frac{\sigma}{\epsilon\omega}\right) \boldsymbol{E} \equiv -i\omega n^2 \epsilon_0 \mu_0 \boldsymbol{E} \tag{5.268}$$

and reveals the squared index of refraction as

$$n^2(\omega) = \frac{\epsilon\mu}{\epsilon_0\mu_0} \left(1 + i\frac{\sigma}{\epsilon\omega}\right). \tag{5.269}$$

The imaginary part of n^2 represents the scattering of light mainly by electrons. At high frequencies in nonmagnetic materials $n^2(\omega) \to 1$, and so Kramers and Kronig applied the Hilbert-transform relations (5.267) to the function $n^2(\omega) - 1$ in order to satisfy condition (5.255). Their relations are

$$\mathrm{Re}(n^2(\omega_0)) = 1 + \frac{2}{\pi} P \int_0^\infty \frac{\omega \, \mathrm{Im}(n^2(\omega))}{\omega^2 - \omega_0^2} \, d\omega \tag{5.270}$$

and

$$\mathrm{Im}(n^2(\omega_0)) = -\frac{2\omega_0}{\pi} P \int_0^\infty \frac{\mathrm{Re}(n^2(\omega)) - 1}{\omega^2 - \omega_0^2} \, d\omega. \tag{5.271}$$

What Kramers and Kronig actually wrote was slightly different from these dispersion relations (5.270 & 5.271). H. A. Lorentz had shown that the index of refraction $n(\omega)$ is related to the forward scattering amplitude $f(\omega)$ for the scattering of light by a density N of scatterers (Sakurai, 1982)

$$n(\omega) = 1 + \frac{2\pi c^2}{\omega^2} N f(\omega). \tag{5.272}$$

They used this formula to infer that the real part of the index of refraction approached unity in the limit of infinite frequency and applied the Hilbert transform (5.267)

$$\mathrm{Re}[n(\omega)] = 1 + \frac{2}{\pi} P \int_0^\infty \frac{\omega' \, \mathrm{Im}[n(\omega')]}{\omega'^2 - \omega^2} \, d\omega'. \tag{5.273}$$

The Lorentz relation (5.272) expresses the imaginary part $\mathrm{Im}[n(\omega)]$ of the index of refraction in terms of the imaginary part of the forward scattering amplitude $f(\omega)$

$$\mathrm{Im}[n(\omega)] = 2\pi (c/\omega)^2 N \mathrm{Im}[f(\omega)]. \tag{5.274}$$

And the **optical theorem** relates $\mathrm{Im}[f(\omega)]$ to the **total cross-section**

$$\sigma_{\mathrm{tot}} = \frac{4\pi}{|\boldsymbol{k}|} \mathrm{Im}[f(\omega)] = \frac{4\pi c}{\omega} \mathrm{Im}[f(\omega)]. \tag{5.275}$$