in the lower half plane. The delta function in the second integral then gives $\pi/2$, so that

$$I = \oint dk \, \frac{e^{ik}}{2i(k + i\epsilon)} + \frac{\pi}{2} = \frac{\pi}{2} \tag{5.229}$$

as stated in (3.109). □

Example 5.36 (The Feynman Propagator) Adding $\pm i\epsilon$ to the denominator of a pole term of an integral formula for a function $f(x)$ can slightly shift the pole into the upper or lower half plane, causing the pole to contribute if a ghost contour goes around the upper half-plane or the lower half-plane. Such an $i\epsilon$ can impose a boundary condition on a Green's function.

The Feynman propagator $\Delta_F(x)$ is a Green's function for the Klein-Gordon differential operator (Weinberg, 1995, pp. 274–280)

$$(m^2 - \Box)\Delta_F(x) = \delta^4(x) \tag{5.230}$$

in which $x = (x^0, \boldsymbol{x})$ and

$$\Box = \triangle - \frac{\partial^2}{\partial t^2} = \triangle - \frac{\partial^2}{\partial (x^0)^2} \tag{5.231}$$

is the four-dimensional version of the laplacian $\triangle \equiv \nabla \cdot \nabla$. Here $\delta^4(x)$ is the four-dimensional Dirac delta function (3.36)

$$\delta^4(x) = \int \frac{d^4q}{(2\pi)^4} \, \exp[i(\boldsymbol{q} \cdot \boldsymbol{x} - q^0 x^0)] = \int \frac{d^4q}{(2\pi)^4} \, e^{iqx} \tag{5.232}$$

in which $qx = \boldsymbol{q} \cdot \boldsymbol{x} - q^0 x^0$ is the Lorentz-invariant inner product of the 4-vectors q and x. There are many Green's functions that satisfy Eq.(5.230). Feynman's propagator $\Delta_F(x)$ is the one that satisfies boundary conditions that will become evident when we analyze the effect of its $i\epsilon$

$$\Delta_F(x) = \int \frac{d^4q}{(2\pi)^4} \, \frac{\exp(iqx)}{q^2 + m^2 - i\epsilon} = \int \frac{d^3q}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dq^0}{2\pi} \, \frac{e^{i\boldsymbol{q}\cdot\boldsymbol{x} - iq^0 x^0}}{q^2 + m^2 - i\epsilon}. \tag{5.233}$$

The quantity $E_{\boldsymbol{q}} = \sqrt{\boldsymbol{q}^2 + m^2}$ is the energy of a particle of mass m and momentum $\boldsymbol{q}$ in natural units with the speed of light $c = 1$. Using this abbreviation and setting $\epsilon' = \epsilon/2E_q$, we may write the denominator as

$$q^2 + m^2 - i\epsilon = \boldsymbol{q} \cdot \boldsymbol{q} - \left(q^0\right)^2 + m^2 - i\epsilon = \left(E_q - i\epsilon' - q^0\right)\left(E_q - i\epsilon' + q^0\right) + \epsilon'^2 \tag{5.234}$$