interval \([-1, 1]\). Let’s promote \(x\) to a complex variable \(z\) and write the square root as \(\sqrt{1-x^2} = -i\sqrt{x^2-1} = -i\sqrt{(z-1)(z+1)}\). As in the last example (5.29), if in both of the square roots we put the cut on the negative (or the positive) real axis, then the function \(f(z) = 1/[(z-k)(-i)\sqrt{(z-1)(z+1)}]\) will be analytic everywhere except along a cut on the interval \([-1, 1]\) and at \(z = k\). The circle \(z = Re^{i\theta}\) for \(0 \leq \theta \leq 2\pi\) is a ghost contour as \(R \to \infty\). If we shrink-wrap this ccw contour around the pole at \(z = k\) and the interval \([-1, 1]\), then we get \(0 = -2I + 2\pi i/((-i)\sqrt{k-1}\sqrt{k+1})\) or

\[
I = -\frac{\pi}{\sqrt{k-1}\sqrt{k+1}}. \tag{5.193}
\]

So if \(k = -2\), then \(I = \pi/\sqrt{3}\), while if \(k = 2\), then \(I = -\pi/\sqrt{3}\).

**Example 5.31 (Contour Integral with a Cut)** Let’s compute the integral

\[
I = \int_0^\infty \frac{x^a}{(x+1)^2} \, dx \tag{5.194}
\]

for \(-1 < a < 1\). We promote \(x\) to a complex variable \(z\) and put the cut on the positive real axis. Since

\[
\lim_{|z| \to \infty} \frac{|z|^{a+1}}{|z+1|^2} = 0, \tag{5.195}
\]

the integrand vanishes faster than \(1/|z|\), and we may add two ghost contours, \(G_+\) counter-clockwise around the upper half-plane and \(G_-\) counter-clockwise around the lower half-plane, as shown in Fig. 5.8.

We add a contour \(C_-\) that runs from \(-\infty\) to the double pole at \(z = -1\), loops around that pole, and then runs back to \(-\infty\); the two long contours along the negative real axis cancel because the cut in \(\theta\) lies on the positive real axis. So the contour integral along \(C_-\) is just the clockwise integral around the double pole which by Cauchy’s integral formula (5.34) is

\[
\oint_{C_-} \frac{z^a}{(z-(-1))^2} \, dz = -2\pi i \left. \frac{dz^a}{dz} \right|_{z=-1} = 2\pi i \, a \, e^{\pi ai}. \tag{5.196}
\]

We also add the integral \(I_-\) from \(\infty\) to 0 just below the real axis

\[
I_- = \int_\infty^0 \frac{(x-i\epsilon)^a}{(x-i\epsilon+1)^2} \, dx = \int_\infty^0 \frac{\exp(a(\ln(x)+2\pi i))}{(x+1)^2} \, dx \tag{5.197}
\]

which is

\[
I_- = -e^{2\pi ai} \int_0^\infty \frac{x^a}{(x+1)^2} \, dx = -e^{2\pi ai} I. \tag{5.198}
\]